Problem solving class 9 (19th Nov)

1. **Dirac equation**

$$(\gamma^\mu p_\mu - M)\,\psi = 0$$

gives back the Klein-Gordon equation after acting with $(\gamma^\mu p_\mu + M)$, if the γ-matrices satisfy the following anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

and $M = mc$. In terms of differential operators,

$$(i\gamma^\mu \partial_\mu - \kappa)\,\psi = 0\,,$$

with $\kappa = \frac{mc}{\hbar}$ being the Compton wavelength.

Trivially, we have $\left(\gamma^0\right)^2 = \mathbb{I}$, $\left(\gamma^i\right)^2 = -\mathbb{I}$, while the traces are zero:

$$\mathrm{Tr}\left(\gamma^0\right) = -\mathrm{Tr}\left(\gamma^0\left(\gamma^i\right)^2\right) = \mathrm{Tr}\left(\gamma^i\gamma^0\gamma^i\right) = \mathrm{Tr}\left(\gamma^0\left(\gamma^i\right)^2\right) = -\mathrm{Tr}\left(\gamma^0\right) = 0 \tag{1}$$

$$\mathrm{Tr}\left(\gamma^i\right) = \mathrm{Tr}\left(\gamma^i\left(\gamma^0\right)^2\right) = -\mathrm{Tr}\left(\gamma^0\gamma^i\gamma^0\right) = -\mathrm{Tr}\left(\gamma^i\left(\gamma^0\right)^2\right) = -\mathrm{Tr}\left(\gamma^i\right) = 0 \tag{2}$$

Furthermore the Dirac matrices' dimension is even. Since $\left(\gamma^0\right)^2 = \mathbb{I}$ and$\left(\gamma^i\right)^2 = -\mathbb{I}$, the eigenvalues are ± 1 and $\pm i$, respectively. Thus, the multiplicity of the eigenvalues $+1$ and -1 (or $+i$ and $-i$) should be the same in order to get a zero trace, meaning that we have an even number of eigenvalues, which equals the dimension of the matrix!

The Dirac group has 32 elements and 17 conjugate classes implying that either we have 16 one dimensional and 1 four dimensional or 12 one dimensional and 5 two dimensional irreducible representations .

2. Let us show that we cannot find four independent 2 dimensional matrices that satisfy the anticommutation relations. Let us take the three Pauli matrices as multiplied by i, satisfying the anticommutation relations:

$$\gamma^j \equiv i\sigma^j,\ \left\{i\sigma^j, i\sigma^k\right\} = -2\delta^{jk} \tag{3}$$

Now if we could find a fourth matrix satisfying the relations in the same way as γ^0, there would be a faithful representation also for 2×2 matrices. Using the fact that the Pauli matrices together with the unit matrix $\sigma^0 \equiv \mathbb{I}_2$ form a basis in the space of 2×2 matrice, we write

$$\gamma^0 = c_0\sigma^0 + c_k\sigma^k$$

and investigate the anticommutators with σ^j,

$$\left\{c_0\sigma^0 + c_k\sigma^k, \sigma^j\right\} = 2c_0\sigma^j - 2c_j = 0 \Leftrightarrow c_0 = 0,\ c_j = 0,\ \forall j = 1,2,3\,, \tag{4}$$

meaning that there cannot exist a fourth 2×2 matrix.

3. **The γ^5 matrix**

Dirac matrices in standard representation:

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} , \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$
$$\beta = \gamma_0 , \quad \alpha_k = \gamma^0\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} , \quad \Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

Important observation is that $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with all the Dirac matrices:

$$\begin{aligned}\gamma^5 &= \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} = -i \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} .\end{aligned}$$

Consequently,

$$\left(\gamma^5\right)^2 = -\mathbb{I}_4 .$$

We can prove this based on the anticommutation relations only,

$$\begin{aligned}\left(\gamma^5\right)^2 &= \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma^0\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= \gamma^2\gamma^3\gamma^2\gamma^3 = \gamma^3\gamma^3 = -\mathbb{I}_4\end{aligned}$$

and also that γ^5 anticommutes e.g. with γ^2

$$\gamma^5\gamma^2 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = \gamma^0\gamma^2\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 \equiv -\gamma^2\gamma^5$$

where we used $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$, $\gamma^0\gamma^0 = \mathbb{I}_4$ and $\gamma^i\gamma^i = -\mathbb{I}_4$ for $i = 1, 2, 3$.

4. **Dirac Hamiltonian**

Multiplying the Dirac equation,

$$\left(\gamma^\mu p_\mu - mc\right)\psi = 0 ,$$

with $c\gamma^0$ and separating the $i\hbar\partial_t$ part, we obtain an equation formally similar to the Schrödinger equation,

$$i\hbar\partial_t\psi = H\psi\left(c\boldsymbol{\alpha}\mathbf{p} + \beta mc^2\right)\psi , \tag{5}$$

with the Dirac-Hamiltonian,

$$H = c\boldsymbol{\alpha}\mathbf{p} + \beta mc^2 .$$

In the presence of electromagnetic field, $A_\mu = (\Phi/c, -\mathbf{A})$, we have to substitute the canonical four momentum p_μ with the kinetic momentum $p_\mu - qA_\mu$,

$$\left(\gamma^\mu(p_\mu - qA_\mu) - mc\right)\psi = 0 , \tag{6}$$

$$\Downarrow$$

$$i\hbar\partial_t\psi = \left(c\boldsymbol{\alpha}(\mathbf{p} - q\mathbf{A}) + q\Phi + \beta mc^2\right)\psi , \tag{7}$$

thus the Dirac Hamiltonian reads as

$$H = c\boldsymbol{\alpha}(\mathbf{p} - q\mathbf{A}) + q\Phi + \beta mc^2 \,. \tag{8}$$

In case of time independent potentials, the time dependence can be separated from the wavefunction, $\psi(\mathbf{r}, t) \to e^{\frac{i}{\hbar}Et}\psi(\mathbf{r})$, leading to the eigenvalue equation of the Dirac Hamiltonian,

$$H\psi = \left(c\boldsymbol{\alpha}(\mathbf{p} - q\mathbf{A}) + q\Phi + \beta mc^2\right)\psi = E\psi \,.$$

For free particles, this simplifies to

$$H\psi = \left(c\boldsymbol{\alpha}\mathbf{p} + \beta mc^2\right)\psi = E\psi$$

$$\Downarrow$$

$$\begin{pmatrix} mc^2 - E & c\boldsymbol{\sigma}\mathbf{p} \\ c\boldsymbol{\sigma}\mathbf{p} & -(E + mc^2) \end{pmatrix}\psi = 0$$

where we separated the Hamiltonian into 2×2 blocks.

Conserved quantity: $\boldsymbol{\Sigma}\mathbf{p} \Rightarrow [H, \boldsymbol{\Sigma}\mathbf{p}] = 0$

$$[H, \boldsymbol{\Sigma}\mathbf{p}] = \begin{pmatrix} mc^2 & c\boldsymbol{\sigma}\mathbf{p} \\ c\boldsymbol{\sigma}\mathbf{p} & -mc^2 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\mathbf{p} \end{pmatrix} - \begin{pmatrix} \boldsymbol{\sigma}\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\mathbf{p} \end{pmatrix}\begin{pmatrix} mc^2 & c\boldsymbol{\sigma}\mathbf{p} \\ c\boldsymbol{\sigma}\mathbf{p} & -mc^2 \end{pmatrix} = 0$$

A little σ gymnastic:

$$\sigma_j\sigma_k = \delta_{jk} + i\epsilon_{jkl}\sigma_l \;, \quad (\boldsymbol{\sigma}\mathbf{p})^2 = \sigma_j p_j \sigma_k p_k = (\mathbb{I}\delta_{jk} + i\epsilon_{jkl}\sigma_l)p_j p_k = p_k p_k \mathbb{I} = \mathbf{p}^2\mathbb{I}$$

Let us look for the solution in the planewave form

$$\psi = e^{i\mathbf{kr}}\begin{pmatrix} \mathsf{u} \\ \mathsf{v} \end{pmatrix}$$

Substituing it into the Dirac equation:

$$\begin{pmatrix} mc^2 - E & c\hbar\boldsymbol{\sigma}\mathbf{k} \\ c\hbar\boldsymbol{\sigma}\mathbf{k} & -(mc^2 + E) \end{pmatrix}\begin{pmatrix} \mathsf{u} \\ \mathsf{v} \end{pmatrix} = 0 \tag{9}$$

which has nontrivial solutions if

$$\det\begin{pmatrix} mc^2 - E & c\hbar\boldsymbol{\sigma}\mathbf{k} \\ c\hbar\boldsymbol{\sigma}\mathbf{k} & -(mc^2 + E) \end{pmatrix} = E^2 - m^2c^4 - c^2\hbar^2\mathbf{k}^2 = 0$$

$$\Downarrow$$

$$E = \pm\sqrt{m^2c^4 + c^2\hbar^2\mathbf{k}^2}$$

Expressing v from the second row of (9):

$$\mathsf{v} = \frac{c\hbar\boldsymbol{\sigma}\mathbf{k}}{mc^2 + E}\mathsf{u}$$

For the positive energy solutions the denominator is greater than $2mc^2$, that is why v is called "small component":

$$(mc^2 - E)\mathsf{u} + c\hbar\boldsymbol{\sigma}\mathbf{k}\,\mathsf{v} = (mc^2 - E)\mathsf{u} + \frac{(c\hbar\boldsymbol{\sigma}\mathbf{k})^2}{mc^2 + E}\mathsf{u} = 0$$

Substituing it to the previous equation, we get indeed

$$m^2c^4 - E^2 + c^2\hbar^2\mathbf{k}^2 = 0\,, \qquad E = \pm\sqrt{m^2c^4 + c^2\hbar^2\mathbf{k}^2}\,.$$

Let w be an arbitrary two component vector, then the orthogonal eigenvectors of the $\boldsymbol{\sigma}\frac{\mathbf{k}}{k}$ helicity matrix can be contructed as

$$\mathsf{u}_\pm = \left(\mathbb{I} \pm \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\mathsf{w}$$

since

$$\boldsymbol{\sigma}\frac{\mathbf{k}}{k}\left(\mathbb{I} \pm \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\mathsf{w} = \pm\underbrace{\left(\mathbb{I} \pm \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\mathsf{w}}_{\text{eigenvector}}\,,$$

$$\mathsf{u}_+^\dagger\mathsf{u}_- = \mathsf{w}^\dagger\left(\mathbb{I} \pm \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\left(\mathbb{I} \mp \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\mathsf{w} = 0\,.$$

So the free solution takes the form for the positive energy sector

$$\mathcal{N}\begin{pmatrix} \mathsf{u}_\pm \\ \pm\frac{c\hbar k}{mc^2+E_+}\mathsf{u}_\pm \end{pmatrix}\,, \quad \mathsf{u}_\pm = \left(\mathbb{I} \pm \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\mathsf{w}\,.$$

It goes in an analogous wway for the negative nergy solutions, where we employ the second row of the eigenvalue equation:

$$\begin{pmatrix} mc^2 - E & c\hbar\boldsymbol{\sigma}\mathbf{k} \\ c\hbar\boldsymbol{\sigma}\mathbf{k} & -(mc^2 + E) \end{pmatrix}\begin{pmatrix} \mathsf{u} \\ \mathsf{v} \end{pmatrix} = 0$$

Expressing now u from the first row:

$$\mathsf{u} = \frac{c\hbar\boldsymbol{\sigma}\mathbf{k}}{E_- - mc^2}\mathsf{v}$$

with $E_- = -\sqrt{(pc)^2 + (mc^2)^2}$, then by similar considerations and choosing the eigenbasis according to the eigenvectors of the helicity matrix we obtain

$$\mathcal{N}\begin{pmatrix} \pm\frac{c\hbar k}{E_- - mc^2}\mathsf{v}_\pm \\ \mathsf{v}_\pm \end{pmatrix}\,, \quad \mathsf{v}_\pm = \left(\mathbb{I} \pm \boldsymbol{\sigma}\frac{\mathbf{k}}{k}\right)\mathsf{w}$$

5. **Time derivative of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$**

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{i}{\hbar}[H, \mathbf{L}]$$

For simplicity, let's consider zero magnetic field $\mathbf{B} = 0$, thus we can take $\mathbf{A} = 0$. Making use that L_i acts simultaneously on the four components of the wavefunction, in fact it shouldbe written as $\mathbb{I}_4 \otimes L_i$, it commutes with $\boldsymbol{\alpha}$ and β,

$$\begin{aligned}[H, L_i] &= [c\alpha_j p_j + q\Phi + \beta mc^2, L_i] \\ &= c\alpha_j[p_j, L_i] + [q\Phi, L_i]\,.\end{aligned}$$

Thus, only the following commutators have to be calculated,

$$[p_j, L_i] = \varepsilon_{ikl}\,[p_j, x_k p_l] = \varepsilon_{ikl}\,[p_j, x_k]\,p_l = \frac{\hbar}{i}\varepsilon_{ikl}\delta_{jk}p_l = \frac{\hbar}{i}\varepsilon_{ijl}p_l$$

$$\Downarrow$$

$$\alpha_j[p_j, L_i] = \frac{\hbar}{i}\varepsilon_{ijl}\alpha_j p_l = \frac{\hbar}{i}(\boldsymbol{\alpha}\times\mathbf{p})_i$$

and

$$\begin{aligned}[q\Phi, L_i] &= -\varepsilon_{ijk}\,[x_j p_k, q\Phi] = -\varepsilon_{ijk}\,x_j\,[p_k, q\Phi] = -\frac{\hbar}{i}\varepsilon_{ijk}\,x_j\,[\partial_k, q\Phi] \\ &= -\frac{\hbar}{i}\varepsilon_{ijk}\,x_j\partial_k\,(q\Phi)\end{aligned}$$

$$\Downarrow$$

$$[q\Phi, \mathbf{L}] = -\frac{\hbar}{i}\left(\mathbf{r}\times\boldsymbol{\nabla}\,(q\Phi)\right)$$

giving the total time-derivative of $\mathbf{L}$,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = c\boldsymbol{\alpha}\times\mathbf{p} + \mathbf{r}\times\mathbf{F}$$

where

$$\mathbf{F} = -\boldsymbol{\nabla}\,(q\Phi)\ .$$

Homework: Prove that the time-derivative of $\mathbf{J} = \mathbf{L} + \mathbf{S}$ with $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\Sigma}$ is

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \frac{i}{\hbar}\,[H, \mathbf{J}] = \mathbf{r}\times\mathbf{F}$$

Dirac group

The four γ^μ as generators of the Dirac group give the following group elements when using the anticommutation relations:

$$\begin{aligned}&\pm\,\mathbb{I},\ \pm\gamma^0,\ \pm\gamma^1,\ \pm\gamma^2,\ \pm\gamma^3 \\ &\pm\,\gamma^0\gamma^1,\ \pm\gamma^0\gamma^2,\ \pm\gamma^0\gamma^3,\ \pm\gamma^1\gamma^2,\ \pm\gamma^1\gamma^3,\ \pm\gamma^2\gamma^3 \\ &\pm\,\gamma^0\gamma^1\gamma^2,\ \pm\gamma^0\gamma^1\gamma^3,\ \pm\gamma^1\gamma^2\gamma^3,\ \pm\gamma^0\gamma^2\gamma^3 \\ &\pm\,\gamma^0\gamma^1\gamma^2\gamma^3,\end{aligned} \tag{10}$$

altogether 32 elements. Now we show that there are 17 conjugate classes:
By definition if $h^{-1}gh = f$, then g and f are in the same conjugate class. Now it is easy to see that by conjugation only the sign of the group elements can be changed, but not the γ^μ matrices in them, that is why each term containing γ^μ matrices are in 2 element conjugate classes with a group element that are -1 times itself. For example using the anticommutaion relations $\gamma^0\gamma^2\gamma^0 = -\gamma^2$, $\left(\gamma^0\right)^{-1} = \gamma^0$. So the 30 group elements with γ^μ matrices form 15 conjugate classes:

$$\begin{aligned}&\pm\,\gamma^0,\ \pm\gamma^1,\ \pm\gamma^2,\ \pm\gamma^3 \\ &\pm\,\gamma^0\gamma^1,\ \pm\gamma^0\gamma^2,\ \pm\gamma^0\gamma^3,\ \pm\gamma^1\gamma^2,\ \pm\gamma^1\gamma^3,\ \pm\gamma^2\gamma^3 \\ &\pm\,\gamma^0\gamma^1\gamma^2,\ \pm\gamma^0\gamma^1\gamma^3,\ \pm\gamma^1\gamma^2\gamma^3,\ \pm\gamma^0\gamma^2\gamma^3 \\ &\pm\,\gamma^0\gamma^1\gamma^2\gamma^3,\end{aligned} \tag{11}$$

while the $-\mathbb{I}$ and $\mathbb{I}$ cannot be transformed into each other by conjugation, as all group elements commute with them and by definition $h^{-1}(\pm\mathbb{I})\,h = h^{-1}h(\pm\mathbb{I}) = \pm\mathbb{I}$, so they form 2 additional one element conjugate classes, altogether giving a total of 17 conjugate classes.