7. gyakorlat (nov. 5)

1. **Solenoid with arbitrary angle dependence**

   Consider a vector potential in cylindrical coordinates:

$$\mathbf{A} = A_\varphi(\varphi,\rho)\,\mathbf{e}_\varphi,\ A_\varphi(\varphi,\rho) = \frac{\Phi g(\varphi)}{\rho},\ A_\rho = 0,\ A_z = 0 \tag{1}$$

   with $\int_0^{2\pi} \mathrm{d}\varphi\, g(\varphi) = 1$.

   The curl of $\mathbf{A}$ is zero for $\rho \neq 0$, rot$\mathbf{A} = 0$, neverthelss its flux is finite:

$$\Phi = \int \mathrm{d}^2\mathbf{f}\,\mathbf{B} = \oint \mathrm{d}\mathbf{r}\,\mathbf{A} = \int_0^{2\pi} \mathrm{d}\varphi\,\rho\frac{\Phi g(\varphi)}{\rho} = \Phi \tag{2}$$

   It is satisfactory for us to consider the angle part of the Hamiltonian as only tha azimuthal angle's momentum will be affected by the vector potential:

$$H = \frac{1}{2m}(p_\varphi - qA_\varphi)^2 \tag{3}$$

   with p_φ corresponding to a fixed radius of a circular motion, $p_\varphi = \frac{\hbar}{ia}\partial_\varphi$:

$$\frac{\hbar^2}{2ma^2}\left(-i\partial_\varphi - \frac{\Phi q}{\hbar}g(\varphi)\right)^2\psi = E\psi \tag{4}$$

   This differential equation is easily solved when first found the eigenfunctions of the operator which is squared without the coefficients from which we can easily express the energy:

$$\left(-i\partial_\varphi - \frac{\Phi q}{\hbar}g(\varphi)\right)\psi = \lambda\psi \Rightarrow E = \frac{\hbar^2}{2ma^2}\lambda^2 \tag{5}$$

   Now the solution/corresponding eigenfunction of the "simplified" eigenvalue differential equation:

$$\partial_\varphi\psi = i\left(\lambda + \frac{\Phi q}{\hbar}g(\varphi)\right)\psi \Rightarrow \psi(\varphi) = Ce^{i\lambda\varphi + \frac{i\Phi q}{\hbar}\int \mathrm{d}\varphi g(\varphi)} \tag{6}$$

   Where the possible values for λ come from the constraint that ψ must be single valued, i.e.: periodic in $\varphi \Rightarrow \psi(0) = \psi(2\pi)$:

$$\lambda 2\pi + \frac{\Phi q}{\hbar}\int_0^{2\pi} \mathrm{d}\varphi\, g(\varphi) = 2n\pi \Rightarrow \lambda_n = n - \frac{\Phi q}{\hbar} \Rightarrow E_n = \frac{\hbar^2}{2ma^2}\left(n - \frac{\Phi}{\Phi_0}\right)^2 \tag{7}$$

   So the lesson is that, although the particle is moving in a regime where there is no magnetic field, the magnetic field that it encircles still influences its dynamics, namely exactly by the fraction of its flux with respect to the fluxoid quantum.

2. **Landau levels in asymmetric gauge**:

   Let the vector potential be

$$\mathbf{A} = \begin{bmatrix} 0 \\ Bx \\ 0 \end{bmatrix} \tag{8}$$

   (a) Write up the Hamiltonian of a particle moving in electromagnetic field!

   The Hamiltonian is simply again just the free particle Hamiltonian with kinetic momentum expressed in terms of the canonical momentum $\mathbf{p} = \frac{\hbar}{i}\nabla$ $\mathbf{K} = \mathbf{p} - q\mathbf{A}$:

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} = \frac{\mathbf{p}^2}{2m} + \frac{B^2q^2}{2m}x^2 - \frac{qB}{m}p_y x = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega_L^2 x^2 - \omega_L p_y x \tag{9}$$

(b) Show that this Hamiltonian commutes with p_y and p_z!

There is neither z nor y dependent part in the Hamiltonian, and as $p_y = \frac{\hbar}{i}\partial_y$, $p_z = \frac{\hbar}{i}\partial_z$ are partial differentiation operators of independent variables we immediately have that $p_y p_i = p_i p_y$, $p_z p_i = p_i p_z$ according to Young's theorem and furthermore $p_y x = p_z x = 0$! So altogether both p_y and p_z commute with all kind of terms in H verifying that

$$[H, p_y] = [H, p_z] = 0 \tag{10}$$

(c) Now as H commutes with p_y and p_z they have a common eigenbasis implying the wave function to be of the form:

$$\phi(x,y,z) = \varphi_{k_y,n}(x)\, e^{ik_y y} e^{ik_z z} \tag{11}$$

where the x dependent part can depend on the k_y quantum number as p_y *does* appear in the Hamiltonian! Wrting it in the Schrödinger equation and exploiting that $\partial_y \leftrightarrow ik_y$, $\partial_z \leftrightarrow ik_z$

$$\begin{aligned}
&\left(-\frac{\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) + \frac{1}{2}m\omega_L^2 x^2 - \omega_L p_y x\right)\varphi_{k_y,n}(x)\, e^{ik_y y} e^{ik_z z} \\
&= \left(-\frac{\hbar^2}{2m}\left(\partial_x^2 - k_y^2 - k_z^2\right) + \frac{1}{2}m\omega_L^2 x^2 - \omega_L \hbar k_y x\right)\varphi_{k_y,n}(x)\, e^{ik_y y} e^{ik_z z} = E_{n,k_y,k_z}\varphi_{k_y,n}(x)\, e^{ik_y y} e^{ik_z z} \\
&\Rightarrow \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_L^2 x^2 - \omega_L \hbar k_y x\right)\varphi_{k_y,n}(x) = E_{n,k_y}\varphi_{k_y,n}(x)
\end{aligned} \tag{12}$$

where $E_{n,k_y,k_z} = E_{n,k_y} - \frac{\hbar^2}{2m}\left(k_y^2 + k_z^2\right)$. Now completing the square, $\frac{1}{2}m\omega_L^2\left(x^2 - 2\frac{\hbar k_y x}{m\omega_L}\right) = \frac{1}{2}m\omega_L^2\left(\left(x - \frac{\hbar k_y}{m\omega_L}\right)^2 - \left(\frac{\hbar k_y}{m\omega_L}\right)^2\right) \equiv \frac{1}{2}m\omega_L^2\left(x - k_y L_H^2\right)^2 - \frac{1}{2}\frac{\hbar^2 k_y^2}{2m}$, where $L_H = \frac{\hbar}{m\omega_L} = \frac{\hbar}{qB}$

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_L^2 (x - x_0)^2\right)\varphi_{n,k_y}(x) = \left(E_{n,k_y,k_z} - \frac{\hbar^2 k_z^2}{2m}\right)\varphi_{n,k_y}(x) \equiv E_{n,k_y}\varphi_{n,k_y}(x) \tag{13}$$

Clearly this is nothing else but a harmonic oscilaltor with $\omega_L = \frac{qB}{m}$ frequency and displacement $x_0 \equiv k_y L_H^2$ and with total energy spectrum and wave- function:

$$E_{n,k_y,k_z} = \hbar\omega_L\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} \tag{14}$$

$$\psi_{n,k_y,k_z}(x,y,z) = \mathcal{N}_n e^{ik_z z} e^{ik_y y}\frac{1}{\sqrt{L_H}} e^{-\frac{1}{2L_H^2}\left(x + k_y L_H^2\right)^2} H_n\left(\frac{x + k_y L_H^2}{L_H}\right) \tag{15}$$

where actually it turned out that the energy does not depend on k_y, $E_{n,k_y,k_z} \equiv E_{n,k_z}$.

Having a system with finite volume of L_x and L_y, then we have for the k_y quantum number the following possible values:

$$k_y = \frac{2\pi}{L_y}m, \quad (m \in \mathbb{N}) \tag{16}$$

While the quantity $k_y L_H^2$ gives the distance between the Landau-orbits with the same energy within a range of size L_x:

$$L_H^2 \frac{2\pi}{L_y} m_{\max} = L_x \Rightarrow m_{\max} = \frac{L_x L_y}{2\pi L_H^2} = \frac{A|q|B}{\hbar} = \frac{\Phi}{\Phi_0} \tag{17}$$

That is, the integer multiple of the fluxoid quantum, which is equal to the degeneracy of the Landau levels.

We can interpret this result in a quasi-classical picture, namely the broadening of the particle in a Landau level is approximately $r \approx \sqrt{2}L_H$, through which we obtain a magnetic flux of

$$\Phi = 2\pi L_H^2 B = \frac{hB}{m\omega_L} = \frac{h}{e} = \Phi_0 \tag{18}$$

meaning that one Landau level, semiclassically corresponds to circular motion through which approximately a fluxoid quantum is pumped through, now in accordance with our previous result, the degeneracy counts "how many" such orbits exist for a given level giving the corresponding multiple of the fluxoid quantums!

3. **Charged particle in uiform magnetic and electric field**

A charged particle is moving in the presence of uniform magnetic and electric field. The magnetic field is parallel to the z axis and the electric field is parallel to the y axis. Using asymmetric gauge $\mathbf{A} = (-By, 0, 0)$ the Hamiltonian can be given as

$$H = \frac{1}{2m}(p_x + qBy)^2 + \frac{p_y^2}{2m} - q\mathcal{E}y$$

Expanding the square of the kinetic momentum operator one can write:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \omega_L p_x y - \mathcal{E}qy + \frac{1}{2}m\omega_L^2 y^2 \ ,$$

where the $\omega_L = \dfrac{qB}{m}$ Larmor frequency has been introduced. The Hamiltonian commutes with the x component of the momentum operator: $[H, p_x] = 0$, consequently, p_x and H have common eigenstates, which can be written in a product form:

$$\psi(x, y) = e^{ik_x x}\varphi(y)$$

Substituting this form into the Schrödinger equation we will arrive at the following simplified Hamiltonian:

$$H = \frac{\hbar k_x^2}{2m} + \frac{p_y^2}{2m} + \omega_L \hbar k_x y - \mathcal{E}qy + \frac{1}{2}m\omega_L^2 y^2$$

In classical mechanics if $Bv = \mathcal{E}$ the force due to the electric field will compensate the magnetic force and the particle will move on a straight line with constant speed. In our case $v_x = \dfrac{\hbar k_x}{m}$ and we get the condition $B\dfrac{\hbar k_x}{m} = \mathcal{E}$. If the previous condition is fulfilled we will get back the Hamiltonian of a simple harmonic oscillator in y direction while we have a free particle in x direction.

Completing the square in the potential energy:

$$\omega_L p_x y - \mathcal{E}qy + \frac{1}{2}m\omega_L^2 y^2 = \frac{1}{2}m\omega_L^2\left(y^2 + 2\frac{\hbar k_x}{m\omega_L}y - 2\frac{\mathcal{E}q}{m\omega_L^2}y\right)$$
$$= \frac{1}{2}m\omega_L^2\left(y + \frac{\hbar k_x}{m\omega_L} - \frac{\mathcal{E}q}{m\omega_L^2}\right)^2 - \frac{\hbar^2 k_x^2}{2m} - \frac{\mathcal{E}^2 q^2}{2m\omega_L^2} + \frac{\hbar k_x}{m\omega_L}\mathcal{E}q \ ,$$

and writing back to the Hamiltonian

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_L^2\left(y + \frac{\hbar k_x}{m\omega_L} - \frac{\mathcal{E}q}{m\omega_L^2}\right)^2 - \frac{\mathcal{E}^2 q^2}{2m\omega_L^2} + \frac{\hbar k_x}{m\omega_L}\mathcal{E}q$$

we will get the Hamiltonian of a harmonic oscillator which is shifted in y direction and its energy is also modified. The energy of the system is

$$E_n(k_x) = \hbar\omega_L\left(n + \frac{1}{2}\right) - \frac{\mathcal{E}^2 q^2}{2m\omega_L^2} + \frac{\hbar k_x}{m\omega_L}\mathcal{E}q$$

We can now observe that with the velocity defiend as $v_x = \frac{\hbar k_x}{m}$ we indeed get back usual harmonic oscillator but with a shift in the energy

$$H\Big|_{v_x = \frac{\hbar k_x}{m}} = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_L^2 y^2 + \frac{1}{2}\frac{\hbar k_x}{m\omega_L}\mathcal{E}q$$

Now as we can see energy is not bounded from below anymore and we see that nevertheless the energy became k_x dependent, as illustrated in the figure below, we will still have infinitely many degeneracies, for a given fix E energy in the limit $L_x \to \infty$. Other important observation is the absence of the $k_x \leftrightarrow -k_x$ symmetry indicating the breakdown of time-reversal-symmetry in the x direcetion!

Degeneracy of the states: The system has continuous spectrum and each levels are infinitly degen-

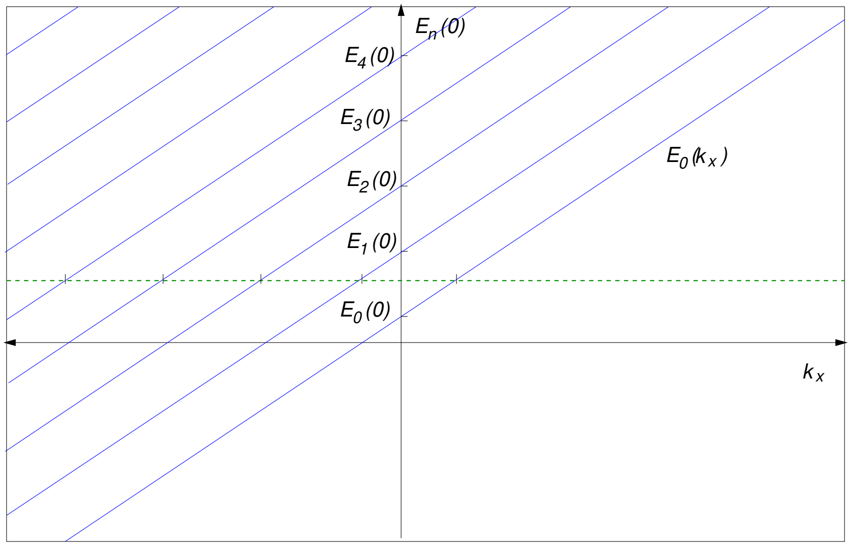

Figure 1: Energy levels of the system. The distance of the dashed line fron the k_X axis represents the energy of the system. The intersections of the dashed line and the energy levels indicated by the straight lines are the states corresponding to the same energy

erate.

4. **Gauge transformation between asymmetric gauges**

Let us give the gauge transformation between the two asymmetric gauges of homogenous $\mathbf{B} = (0, 0, B)$ magnetic field, $\mathbf{A} = (-By, 0, 0)$ and $\mathbf{A}' = (0, Bx, 0)$.

Solution:

As calculated above and in the lecture we have, of course, the same energy for the tow above gauges and the follwoing wave-functions:

$$E_{n,k_y,k_z} = E_{n,k_x,k_z} = \hbar\omega_L\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} \tag{19}$$

$$\psi_{n,k_x,k_z}(\mathbf{r}) = \mathcal{N}_n \exp(ik_x x)\exp(ik_z z)\,\frac{1}{\sqrt{L_H}}\exp\left(-\frac{1}{2L_H^2}\left(y + k_x L_H^2\right)^2\right) H_n\left(\frac{y + k_x L_H^2}{L_H}\right) \tag{20}$$

$$\psi'_{n,k_y,k_z}(\mathbf{r}) = \mathcal{N}_n \exp(ik_y y)\exp(ik_z z)\,\frac{1}{\sqrt{L_H}}\exp\left(-\frac{1}{2L_H^2}\left(x - k_y L_H^2\right)^2\right) H_n\left(\frac{x - k_y L_H^2}{L_H}\right) \tag{21}$$

(22)

The two gauges are realted two eac other by the gauge function $\Lambda = Bxy$ by the usual relation:

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda \tag{23}$$

Now the transformed wave function and its expansion in the basis of the $\mathbf{A}'$ gauge:

$$\exp\left(-\frac{iq}{\hbar}\Lambda\right)\psi_{n,k_x,k_z}(\mathbf{r}) = \int \mathrm{d}k_y\, c_{n,k_x,k_y}\,\psi'_{n,k_y,k_z}(\mathbf{r}) \tag{24}$$

with the c_{n,k_y,k_z} being the expansion coefficients in the $\mathbf{A}$ basis:

$$\begin{aligned} c_{n,k_x,k_y} &= \int \mathrm{d}^3\mathbf{r}\, \psi'_{n,k_y,k_z}(\mathbf{r})^* \exp\Big(-\frac{ie}{\hbar}\Lambda\Big)\, \psi_{n,k_x,k_z}(\mathbf{r}) \\ &= \frac{\mathcal{N}_n^2}{L_H^2} \int \mathrm{d}x\mathrm{d}y\, \exp\Big(-\frac{1}{2L_H^2}\big(x-k_yL_H^2\big)^2\Big)\, H_n\Big(\frac{x-k_yL_H^2}{L_H}\Big) \exp\Big(-\frac{1}{2L_H^2}\big(x-k_yL_H^2\big)^2\Big)\, H_n\Big(\frac{x-k_yL_H^2}{L_H}\Big) \\ &\times \exp\big(ik_yy - ik_xx - ixy/L_H^2\big) \end{aligned} \tag{25}$$

Introducing new varaibles for sake of convenience, $\xi_x = x/L_H$, $\xi_y = y/L_H$, $q_x = k_xL_H$, $q_y = yL_H$, with which the above integral comes to the form:

$$\begin{aligned} c_{n,k_x,k_y} &= \mathcal{N}_n^2 L_H \int \mathrm{d}\xi_x \mathrm{d}\xi_y\, \exp(-iq_y(\xi_y+q_x) - iq_x(\xi_x-q_y) - i(\xi_x-q_y)(\xi_y+q_x)) \\ &\times \exp\Big(-\frac{1}{2}\big(\xi_x^2+\xi_y^2\big)^2\Big)\, H_n(\xi_x)\, H_n(\xi_y) \\ &= \mathcal{N}_n^2 L_H \exp(-iq_xq_y) \int \mathrm{d}\xi_x \mathrm{d}\xi_y\, \exp(-i\xi_x\xi_y) \exp\Big(-\frac{1}{2}\xi_x^2 - \frac{1}{2}\xi_y^2\Big)\, H_n(\xi_x)\, H_n(\xi_y) \end{aligned} \tag{26}$$

Now if could calculate this coefficient we would have expansion coefficients and so linear transformation "rule" between the eigenfaunctions of the two gauges!

Statement:

$$\sqrt{2\pi} \exp\Big(-\frac{1}{2}\xi_y^2\Big)\, H_n(\xi_y) = \int \mathrm{d}x\, \exp(-i\xi_x\xi_y) \exp\Big(-\frac{1}{2}\xi_x^2\Big)\, H_n(\xi_x) \tag{27}$$

For proving it we write up the wave-function of the harmonic oscillator in coordiante representation:

$$\psi_n(x) = \mathcal{N}_n \frac{1}{\sqrt{x_0}} \exp\Big(-\frac{1}{2}\frac{x^2}{x_0^2}\Big)\, H_n\Big(\frac{x}{x_0}\Big), \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \tag{28}$$

The we also write it up in momentum representaion where one read out the modified constants from the corresponding Schrödinger equation:

$$\Big(\frac{p^2}{2m} - \frac{1}{2}\hbar^2 m\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2}\Big)\, \tilde{\psi}(p) = E\tilde{\psi}(p) \tag{29}$$

$$\tilde{\psi}(p) = \mathcal{N}_n \frac{1}{\sqrt{p_0}} \exp\Big(-\frac{1}{2}\frac{p^2}{p_0^2}\Big)\, H_n\Big(\frac{p}{p_0}\Big) \tag{30}$$

Now as we know that in coordiante representaion $\langle x|p\rangle = \frac{\exp\left(\frac{i}{\hbar}px\right)}{\sqrt{\hbar}}$ we have the expansion:

$$\mathcal{N}_n \frac{1}{\sqrt{p_0}} \exp\Big(-\frac{1}{2}\frac{p^2}{p_0^2}\Big)\, H_n\Big(\frac{p}{p_0}\Big) = \int \mathrm{d}x \frac{\exp\big(\frac{i}{\hbar}px\big)}{\sqrt{\hbar}} \frac{1}{\sqrt{x_0}} \exp\Big(-\frac{1}{2}\frac{x^2}{x_0^2}\Big)\, H_n\Big(\frac{x}{x_0}\Big) \tag{31}$$

Now with the new variables again, $\xi_x = \frac{x}{x_0}$, $\xi_y = \frac{p}{p_0}$ we have the desired identity:

$$\exp\Big(-\frac{1}{2}\xi_y^2\Big)\, H_n(\xi_y) = \sqrt{\frac{x_0p_0}{\hbar}} \int \mathrm{d}x\, \exp(-i\xi_x\xi_y) \exp\Big(-\frac{1}{2}\xi_x^2\Big)\, H_n(\xi_x) \tag{32}$$

Giving the identity of the statement and yielding the relation between the eignebasis of the two gauges:

$$c_{n,k_x,k_y} = \sqrt{2\pi} L_H \exp(-iq_xq_y)\, \mathcal{N}_n^2 \int \mathrm{d}\xi\, \exp\big(-\xi^2\big)\, H_n(\xi) = \sqrt{2\pi} L_H \exp\big(-ik_xk_y/L_H^2\big) \tag{33}$$

giving the final relation:

$$\psi_{n,k_x,k_z}(\mathbf{r}) = \exp\!\left(\frac{ie}{\hbar}\Lambda\right) \int \mathrm{d}k_y \, \sqrt{2\pi} L_H \exp\!\left(-ik_x k_y / L_H^2\right) \psi'_{n,k_y,k_z}(\mathbf{r})$$
$$\left\langle \psi'_{n,k_y,k_z} | \psi_{n,k_x,k_z} \right\rangle = \exp\!\left(\frac{ie}{\hbar}\Lambda\right) \sqrt{2\pi} L_H \exp\!\left(-ik_x k_y / L_H^2\right) \tag{34}$$