5. gyakorlat (okt. 8)

1. Time-evolution pictures:

   (a) **Schrödinger-picture:**

   Wave functions are time-evolved by the Hamiltonian

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1}$$

$$|\psi(t)\rangle = U(t,0)|\psi(0)\rangle = U(t,0)|\psi(0)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle \tag{2}$$

   where the time-evolution operator takes a simple form of $\mathcal{T}e^{-\frac{i}{\hbar}\int_0^t \mathrm{d}t' H(t')}$ in case of time-independent Hamiltonians.

   (b) **Heisenberg-picture:**

   Wave-function remains in its initial state and instead operators are "rotated"/time-evolved according to

$$|\psi^H(t)\rangle = U^\dagger(t,0)|\psi(t)\rangle = |\psi(0)\rangle \tag{3}$$

$$A^H(t) = U^\dagger(t,0)AU(t,0) \tag{4}$$

$$\frac{\mathrm{d}A^H(t)}{\mathrm{d}t} = \frac{i}{\hbar}\left[H^H(t), A^H(t)\right] + \frac{\partial A^H(t)}{\partial t} \tag{5}$$

   One can easily verify that so the time evolution of matrix elements of operators are unchanged!

$$\begin{aligned}\frac{\mathrm{d}\langle A^H(t)\rangle}{\mathrm{d}t} &= \langle\psi(t)|\frac{i}{\hbar}[H,A] + \partial_t A|\psi(t)\rangle = \langle\psi(0)|U^\dagger(t,0)\frac{i}{\hbar}[H,A]\,U(t,0) + \partial_t A^H(t)|\psi(0)\rangle \\ &= \langle\psi(0)|\frac{i}{\hbar}\left[H^H(t), A^H(t)\right] + \partial_t A^H(t)|\psi(0)\rangle\end{aligned} \tag{6}$$

   (c) **Interaction/Dirac-picture:**

   Now wave-functions are time-evolved only by the potential part of the Hamiltonian $H(t) = H_0 + V(t)$, while operators evolve in time with respect to the free Hamiltonian, H_0 with the time-evolution operator, $U(t,0) = e^{-\frac{i}{\hbar}H_0 t}$:

$$|\psi^D(t)\rangle = U^\dagger(t,0)|\psi(t)\rangle \tag{7}$$

$$A^D(t) = U^\dagger(0,t)AU(t,0) \tag{8}$$

$$i\hbar\frac{\partial}{\partial t}|\psi^D(t)\rangle = V^D(t)|\psi^D(t)\rangle \tag{9}$$

   Similarly matrix-elements' time-evolution again remains unchanged in the Dirac-picture, as it should!

   (d) **Time-dependent perturbation theory:**

   Let us first approximate the time-ordered exponential up to first order in the perturbation operator in Dirac-picture, in $V^D(t)$ and compute the wave- function up to first order in Dirac picture again:

$$U^D(t,0) = \mathcal{T}e^{-\frac{i}{\hbar}\int_0^t \mathrm{d}t' H(t')} \approx I - \frac{i}{\hbar}\int_0^t \mathrm{d}t' V^D(t') = I - \frac{i}{\hbar}\int_0^t \mathrm{d}t' e^{\frac{i}{\hbar}H_0 t'}V(t')e^{-\frac{i}{\hbar}H_0 t'} \tag{10}$$

$$|\psi^{D(1)}(t)\rangle = |\varphi\rangle - \frac{i}{\hbar}\int_0^t \mathrm{d}t' e^{\frac{i}{\hbar}H_0 t'}V(t')e^{-\frac{i}{\hbar}H_0 t'}|\varphi\rangle \tag{11}$$

Now let us transform back to Schrödinger picture:

$$|\psi^{S(1)}(t)\rangle = e^{-\frac{i}{\hbar}H_0 t}|\psi^{D(1)}(t)\rangle = e^{-\frac{i}{\hbar}H_0 t}|\varphi\rangle - \frac{i}{\hbar}e^{-\frac{i}{\hbar}H_0 t}\int_0^t \mathrm{d}t' e^{\frac{i}{\hbar}H_0 t'}V(t')e^{-\frac{i}{\hbar}H_0 t'}|\varphi\rangle \quad (12)$$

Now let us choose $|\varphi\rangle \equiv |k\rangle$ some eigenstate of $H_0 = \sum_n \varepsilon_n |n\rangle\langle n|$ with ε_n being the eigenenergies and $|n\rangle$ the eigenstates and with this we have $e^{\pm\frac{i}{\hbar}H_0 t} = \sum_n e^{\pm\frac{i}{\hbar}\varepsilon_n t}|n\rangle\langle n|$, inserting the identity, as $I = \sum_n |n\rangle\langle n|$

$$\begin{aligned} |\psi^{S(1)}(t)\rangle &= e^{-\frac{i}{\hbar}H_0 t}|k\rangle - \frac{i}{\hbar}\sum_n e^{-\frac{i}{\hbar}H_0 t}\int_0^t \mathrm{d}t' e^{\frac{i}{\hbar}H_0 t'}|n\rangle\langle n|V(t')e^{-\frac{i}{\hbar}H_0 t'}|k\rangle \\ &= e^{-\frac{i}{\hbar}\varepsilon_k t}|k\rangle - \frac{i}{\hbar}\sum_n e^{-i\varepsilon_n t}|n\rangle\int_0^t \mathrm{d}t' e^{\frac{i}{\hbar}\omega_{nk}t'}V_{nk}(t') \end{aligned} \quad (13)$$

with $\omega_{nk} = \frac{\varepsilon_n - \varepsilon_k}{\hbar}$ and $V_{nk}(t) = \langle n|V(t)|k\rangle$. From this one can immediately read out the expansion coefficients:

$$|\psi^{S(1)}(t)\rangle = \sum_n c_n^{(1)}(t)e^{-\frac{i}{\hbar}\varepsilon_n t}|n\rangle \quad (14)$$

with the coefficients given thus by:

$$c_k^{(1)}(t) = 1 - \frac{i}{\hbar}\int_0^t \mathrm{d}t' V_{kk}(t') \quad (15)$$

$$c_{n\neq k}^{(1)}(t) = -\frac{i}{\hbar}\int_0^t \mathrm{d}t' e^{i\omega_{nk}t'}V_{nk}(t') \quad (16)$$

2. **Determine the mean displacement, $\langle\psi(t)|x|\psi(t)\rangle$, in case of a free particle moving in constant electric field using the Heisenberg picture description in case of an initial Gaussian wave-packet:**

$$H = \frac{p^2}{2m} - \mathcal{E}qx, \quad i\hbar\partial_t\psi = H\psi \quad (17)$$

Wave packet in electric field, $\psi(0,x) = \frac{1}{\sqrt{x_0\sqrt{\pi}}}e^{-\frac{x^2}{2x_0^2}}$. Knowing the solution of the eigenvalue equation $H\varphi_n = E_n\varphi_n$ we can expand the wave-function started from the initial wave packet as:

$$\psi(t) = \sum_n c_n e^{-\frac{i}{\hbar}Et}\varphi_n, \quad c_n = \langle\varphi_n|\psi(0)\rangle \quad (18)$$

Let us introduce the length scale $x_0 = \left(\frac{\hbar^2}{2m\mathcal{E}q}\right)^{\frac{1}{3}}$ with which we can write the Schrödinger equation as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} - \xi\psi = 0 \quad (19)$$

with $\xi = \frac{x}{x_0} + \frac{E}{x_0\mathcal{E}q}$. The solutions of this equation are the Airy functions, $\psi = \mathrm{Ai}\left(\frac{x}{x_0} + \frac{E}{x_0\mathcal{E}q}\right)$. These solutions form a continuous basis, with continuously many eigenvalues, as there is no potential gap just a potential with constant slope, with infinitely many possible eigenenergies, or with other words no periodic orbits classically. So the time-dependent solution is given by the integral, i.e.: the summation over the continuously many energy eigenstates, reads

$$\psi(t,x) = \int_{-\infty}^{\infty}\mathrm{d}E\, c(E)e^{-\frac{i}{\hbar}Et}\mathrm{Ai}\left(\frac{x}{x_0} + \frac{E}{x_0\mathcal{E}q}\right) \quad (20)$$

with the expansion coefficients defined as:

$$c(E) = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{Ai}\left(\frac{x}{x_0} + \frac{E}{x_0 \mathcal{E} q}\right) \frac{1}{\sqrt{x_0\sqrt{\pi}}} e^{-\frac{x^2}{2x_0^2}} \tag{21}$$

Now we see that in terms of these complicated wave functions we can tell, in theory, in terms of horribly complicated integrals the average displacement, $\langle\psi(t)|x|\psi(t)\rangle$. Nevertheless we come around the problem by the Heisenberg time-evolution picture, considering the operators rather than the wave functions evolving in time, that is it is satisfactory only to consider the initial wave-packet and calculate the time-evolved/transformed operator with the time-evolution operator:

$$x(t) = e^{\frac{i}{\hbar}Ht} x e^{-\frac{i}{\hbar}Ht} \tag{22}$$

the expectation value of which in the initial state gives the same result as the "usual" coordinate operator with the time-evolved states, which in an operator language reads as $|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle$

$$\langle\psi(0)e^{\frac{i}{\hbar}Ht}|x|e^{-\frac{i}{\hbar}Ht}\psi(0)\rangle = \langle\psi(0)|e^{\frac{i}{\hbar}Ht} x e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle \tag{23}$$

Now the only thing to calculate is the transformed coordinate operator, then we only need to calculate a Gaussian integral with this transformed coordinate operator, which can be calculated via the Haussdorf expansion:

$$e^{\frac{i}{\hbar}Ht} x e^{-\frac{i}{\hbar}Ht} = x + \frac{i}{\hbar}\left[H, x\right] t + \left(\frac{i}{\hbar}\right)^2 \left[H, \left[H, x\right]\right] t^2/2 + \ldots \tag{24}$$

Now fortunately the series terminates at the quadratic term as $[H, x] = \frac{\hbar}{i}\frac{p}{m}$, $[H, [H, x]] = \frac{\hbar}{im}[H, p] = \left(\frac{\hbar}{i}\right)^2 \frac{\mathcal{E}q}{m}$, which is just a number, commuting with H and making vanish the fourth term!
So now we are in the position to evaluate the emerging integrals:

$$\langle\psi(0)|e^{\frac{i}{\hbar}Ht} x e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle = \frac{1}{x_0\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{x^2}{2x_0^2}} \left(x + \frac{p}{m}t + \frac{\mathcal{E}q}{2m}t^2\right) e^{-\frac{x^2}{2x_0^2}} = \frac{\mathcal{E}q}{2m}t^2 \tag{25}$$

In fact we were facig a very easy case of Gaussian integrals as both $\sim xe^{-\frac{x^2}{x_0^2}}$ and $\sim e^{-\frac{x^2}{2x_0^2}} \partial_x e^{-\frac{x^2}{2x_0^2}}$ are odd functions integrated over a symmetrical region, yielding zero.

3. **Determine the ladder operators in Heisenberg picture for the one-dimensional harmonic oscillator and show that in coherent states the $\Delta x \Delta p$ prdouct is a constant!**

The Hamiltonian, as usual, $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$, which being time-independent yields the $e^{-\frac{i}{\hbar}Ht}$ time-evolution operator. For this we make use of the usual commutation relations of the ladder operators, $\left[a, a^\dagger\right] = 1 \rightarrow [H, a] = -\hbar\omega a$, $\left[H, a^\dagger\right] = \hbar\omega a^\dagger$, by which we can again conclude that the Haussdorff series goes as:

$$e^{-\frac{i}{\hbar}Ht} a e^{\frac{i}{\hbar}Ht} = a + \frac{i}{\hbar}(-\hbar\omega)at + \left(\frac{i}{\hbar}\right)^2 \frac{(-\hbar\omega)^2}{2!}at^2 + \cdots = ae^{-i\omega t} \tag{26}$$

Similarly for $a^\dagger$:

$$e^{-\frac{i}{\hbar}Ht} a e^{\frac{i}{\hbar}Ht} = a^\dagger + \frac{i}{\hbar}(\hbar\omega)a^\dagger t + \left(\frac{i}{\hbar}\right)^2 \frac{(\hbar\omega)^2}{2!}a^\dagger t^2 + \cdots = a^\dagger e^{i\omega t} \tag{27}$$

Coherent states: eigenstates of ladder operators:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{28}$$

Now let us calculate the variances' time-evolution $\langle\Delta x\rangle_\alpha^2 = \langle\alpha|x^2(t)|\alpha\rangle - \langle\alpha|x(t)|\alpha\rangle^2$ and $\langle\Delta p\rangle_\alpha^2 = \langle\alpha|p^2(t)|\alpha\rangle - \langle\alpha|p(t)|\alpha\rangle^2$, with $x = \frac{x_0}{\sqrt{2}}(a+a^\dagger) \to x(t) = \frac{x_0}{\sqrt{2}}\left(ae^{-i\omega t} + a^\dagger e^{i\omega t}\right),\ x_0 = \sqrt{\frac{\hbar}{m\omega}},\ p = i\frac{p_0}{\sqrt{2}}(a^\dagger - a) \to p(t) = i\frac{p_0}{\sqrt{2}}\left(a^\dagger e^{i\omega t} - ae^{-i\omega t}\right),\ p_0 = \sqrt{m\hbar\omega}$

$$\langle\alpha\,|x(t)|\,\alpha\rangle^2 = \frac{x_0^2}{2}\left\langle\alpha\left|(ae^{-i\omega t} + a^\dagger e^{i\omega t})\right|\alpha\right\rangle^2 = 2x_0^2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)^2$$

$$\langle\alpha\,|p(t)|\,\alpha\rangle^2 = -\frac{p_0^2}{2}\left\langle\alpha\left|(a^\dagger e^{i\omega t} - ae^{-i\omega t})\right|\alpha\right\rangle^2 = 2p_0^2\mathrm{Im}\left(\alpha e^{-i\omega t}\right)^2$$

$$\left\langle\alpha\left|x^2(t)\right|\alpha\right\rangle = \frac{x_0^2}{2}\left\langle\alpha\left|1 + 2a^\dagger a + a^2e^{-2i\omega t} + \left(a^\dagger\right)^2 e^{2i\omega t}\right|\alpha\right\rangle = \frac{x_0^2}{2}\left(1 + 2|\alpha|^2 + 2\mathrm{Re}\left(\alpha^2 e^{-i2\omega t}\right)\right)$$

$$\left\langle\alpha\left|p^2(t)\right|\alpha\right\rangle = -\frac{p_0^2}{2}\left\langle\alpha\left|-1 - 2a^\dagger a + a^2e^{-2i\omega t} + \left(a^\dagger\right)^2 e^{2i\omega t}\right|\alpha\right\rangle = \frac{p_0^2}{2}\left(1 + 2|\alpha|^2 - 2\mathrm{Re}\left(\alpha^2 e^{-i2\omega t}\right)\right)$$

$$\left\langle\alpha\left|\delta x^2(t)\right|\alpha\right\rangle = \left\langle\alpha\left|x^2(t)\right|\alpha\right\rangle - \langle\alpha\,|x(t)|\,\alpha\rangle^2 = \frac{x_0^2}{2} + x_0^2\left(|\alpha|^2 + \mathrm{Re}\left(\alpha^2e^{-i2\omega t}\right) - 2\mathrm{Re}\left(\alpha e^{-i\omega t}\right)^2\right)$$

$$= \frac{x_0^2}{2} + x_0^2\left(|\alpha|^2 - \mathrm{Re}\left(\alpha e^{-i\omega t}\right)^2 - \mathrm{Im}\left(\alpha e^{-i\omega t}\right)^2\right) = \frac{x_0^2}{2} - x_0^2\left(|\alpha|^2 - |\alpha e^{-i\omega t}|^2\right) = \frac{x_0^2}{2}$$

$$\left\langle\alpha\left|\delta p^2(t)\right|\alpha\right\rangle = \left\langle\alpha\left|p^2(t)\right|\alpha\right\rangle - \langle\alpha\,|p(t)|\,\alpha\rangle^2 = \frac{p_0^2}{2} + p_0^2\left(|\alpha|^2 - \mathrm{Re}\left(\alpha^2e^{-i2\omega t}\right) - 2\mathrm{Im}\left(\alpha e^{-i\omega t}\right)^2\right)$$

$$= \frac{p_0^2}{2} + p_0^2\left(|\alpha|^2 - \mathrm{Im}\left(\alpha e^{-i\omega t}\right)^2 - \mathrm{Re}\left(\alpha e^{-i\omega t}\right)^2\right) = \frac{p_0^2}{2}$$

From where we can conclude that time-independently the product of the deviations reads as:

$$\langle\Delta x\rangle_\alpha\langle\Delta p\rangle_\alpha = \frac{x_0p_0}{2} = \frac{\hbar}{2} \tag{29}$$

Homework:

Consider the Hamiltonian of a particle moving in constant electric field:

$$H = \frac{p^2}{2m} - \mathcal{E}qx \tag{30}$$

Repeat the calculation of the 2. exercise but with a particle starting from a coherent state of a harmonic oscillator:

$$|\psi(0)\rangle = e^{-\frac{|\alpha|^2}{2} + \alpha a^\dagger}|0\rangle \tag{31}$$

with the usual relations, $x = \frac{x_0}{\sqrt{2}}\left(a + a^\dagger\right),\ x_0 = \sqrt{\frac{\hbar}{m\omega}}$ and $p = i\frac{p_0}{\sqrt{2}}\left(a^\dagger - a\right),\ p_0 = \sqrt{m\omega\hbar}$. There are two ways to do that:

(a) First expand the exponent to get $\psi(x,0) = \sqrt{\frac{m\omega}{\hbar\sqrt{\pi}}}e^{-\frac{(x-\langle x\rangle)^2}{2x_0^2} + i\frac{\langle p\rangle x}{\hbar}}$ where $\langle p\rangle = \sqrt{\frac{m\hbar\omega}{2}}\mathrm{Re}\alpha$ in a coherent state with α eigenvalue. And with this calculate the above integral (25) with $x(t) = e^{\frac{i}{\hbar}Ht}xe^{-\frac{i}{\hbar}Ht}$ transformed into Heisenberg picture!

(b) Or in a more sofisticated way use our knowledge about the ladder operators in Heisenberg picture, $a(t) = ae^{-i\omega t},\ a^\dagger(t) = a^\dagger e^{i\omega t}$.

Hints: Use the Heisenberg picture form of $x(t)$, derived in the 2. exercise, and write it in terms of the ladder operators, then calculate tha action of a and $a^\dagger$ on a coherent state, as dictated by its definition.