3. gyakorlat (szept. 57.)

1. Determine the following scattering amplitudes in first order Born approximation

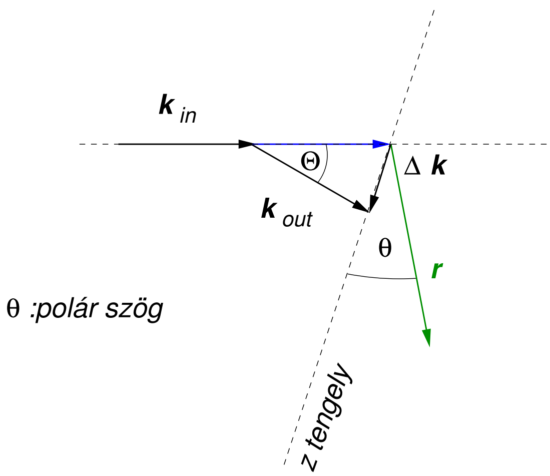
   a.) Yukawa potential: $V(r) = ke^2 \dfrac{e^{-\alpha r}}{r}$,

   b.) Gaussian potential: $V(r) = V_0 e^{-\frac{r^2}{2r_0^2}}$.

   Scattering amplitude in first order Born approximation:

$$f(\vartheta, \varphi) = -\frac{m}{2\pi\hbar^2} \int V(\mathbf{r}) e^{i\mathbf{qr}} d^3r$$

   For spherical potentials:



$$q = \Delta k = 2k \sin\left(\frac{\Theta}{2}\right)$$

$$\int V(\mathbf{r}) e^{i\mathbf{qr}} d^3r = \int V(r) e^{iqr\cos(\theta)} r^2 dr \sin(\theta) d\vartheta d\varphi$$
$$= 2\pi \int V(r) e^{iqr\cos(\theta)} r^2 dr \sin(\theta) d\theta$$
$$= 2\pi \int_0^\infty r^2 dr \int_{-1}^1 dz V(r) e^{iqrz}$$
$$= \frac{4\pi}{q} \int_0^\infty V(r) \sin(qr) r dr$$

   a.) Use the above formula for the Yukawa potential! Determine the total cross section as well! What happens if $\alpha \to 0$?

   So the task is easy, we just have to calculate the integral (for sake of simplicity without prefactors):

$$\int_0^\infty \mathrm{d}r\, e^{-\alpha r} \sin(qr) = \frac{1}{2i} \int_0^\infty \mathrm{d}r\, e^{(-\alpha+iq)r} - e^{(-\alpha-iq)r} = \frac{1}{2i}\left(\frac{1}{-\alpha+iq} - \frac{1}{-\alpha-iq}\right) = \frac{q}{\alpha^2+q^2} \qquad (1)$$

   From where the scattering amplitude and for notational convenience $ke^2 = \frac{e^2}{4\pi\varepsilon_0}$:

$$f(\theta, \varphi) = -\frac{m}{\hbar^2} \frac{e^2}{2\pi\varepsilon_0(\alpha^2+q^2)}$$

   From where the differential cross section with the prefactors:

$$\sigma(q) = |f(\theta, \varphi)|^2 = \frac{m^2 e^4}{4\hbar^4 \pi^2 \varepsilon_0^2 (\alpha^2+q^2)^2}$$

   Now let us integrate it with respect to $\mathrm{d}\Omega = \mathrm{d}\varphi\, \mathrm{d}\theta \sin\theta$, (where as we did not need the θ variable used in the $\int \mathrm{d}^3\mathbf{r}$ integrals, we can without any further complication introduce the more convenient θ notation instead of Θ for dΩ angle integral) using that $q = 4k^2 \sin^2(\theta/2)$

$$\sigma_{\text{tot}} = \frac{m^2 e^4}{4\hbar^4 \pi^2 \varepsilon_0^2} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \frac{1}{\left(\alpha^2 + 4k^2 \sin^2(\theta/2)\right)^2} \to \frac{1}{\alpha^4} \int_{-1}^{1} \mathrm{d}x \, \frac{1}{\left(1 + 2(k/\alpha)^2 - 2(k/\alpha)^2 x\right)^2}$$
$$= \frac{1}{2\alpha^2 k^2} \left(1 - \frac{1}{1 + 4(k/\alpha)^2}\right) = \frac{2}{\alpha^2 (4k^2 + \alpha^2)} \to \frac{m^2 e^4}{2\hbar^4 \pi^2 \varepsilon_0^2} \frac{1}{\alpha^2 (4k^2 + \alpha^2)} \tag{2}$$

Now let us investigate the case of $\alpha \to 0$ for the amplitude, corrsponding to the case of Coulomb potential:

$$\lim_{\alpha \to 0} f(\theta, \varphi) = -\lim_{\alpha \to 0} \frac{m}{\hbar^2} \frac{e^2}{2\pi\varepsilon_0 (4\alpha^2 + q^2)} = -\frac{me^2}{2\pi\varepsilon_0} \frac{1}{q^2}$$

Now the cross section:

$$\lim_{\alpha \to 0} \sigma(q, \alpha) = \frac{m^2 e^4}{4\pi^2 \varepsilon_0^2} \frac{1}{q^4}$$

b.) In case of the Gaussian potential integrate directly!

$$\int V_0 e^{-\frac{r^2}{2r_0^2}} e^{i\mathbf{qr}} d^3 r = V_0 \int_{-\infty}^{\infty} e^{-\frac{x^2}{2r_0^2}} e^{iq_x x} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2r_0^2}} e^{iq_y y} dy \int_{-\infty}^{\infty} e^{-\frac{z^2}{2r_0^2}} e^{iq_z z} dz$$

$$\int_{-\infty}^{\infty} e^{-\alpha(x - ia)^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

As the contour can be closed in the complex z plane such that the line $z - ia$ is connected to the real axes $z \in [-\infty, \infty]$ by z being purely imaginary with infinite imaginary part, annulalting the integral along these lines as the enclosed area contains no singularity, the integral along the real axes and the one shifted by ia gives the same result but with different signs!
So it is satisfactory to compute only one Gaussian integral gives:

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{x^2}{2r_0^2} + iq_x x} = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{1}{2r_0^2}(x - iq_x r_0^2)^2 - \frac{q_x^2 r_0^2}{2}} = e^{-\frac{q_x^2 r_0^2}{2}} \sqrt{2\pi r_0^2}$$

From here we can easily get the scattering amplitude:

$$f(\theta, \varphi) = -\frac{m}{2\pi\hbar^2} V_0 e^{-\frac{q^2 r_0^2}{2}} (2\pi r_0^2)^{3/2} = -\frac{mV_0 \sqrt{2\pi} r_0^3}{\hbar^2} e^{-\frac{q^2 r_0^2}{2}}$$

The differential cross section:

$$\sigma(q) = |f(\theta, \varphi)|^2 = \frac{m^2 V_0^2 2\pi r_0^6}{\hbar^4} e^{-q^2 r_0^2}$$

Now again compute the total cross section:

$$\sigma_{\text{tot}} = 2\pi \frac{m^2 V_0^2 2\pi r_0^6}{\hbar^4} \int_0^{\pi} \mathrm{d}\theta \, \sin\theta e^{-2k^2 \sin^2(\theta/2) r_0^2} = \frac{4m^2 V_0^2 \pi^2 r_0^6}{\hbar^4} \int_{-1}^{1} \mathrm{d}x \, e^{-2k^2 r_0^2 (1-x)}$$
$$= \frac{4m^2 V_0^2 \pi^2 r_0^6}{\hbar^4} e^{-2k^2 r_0^2} \int_{-1}^{1} \mathrm{d}x \, e^{2k^2 r_0^2 x} = \frac{2m^2 V_0^2 \pi^2 r_0^4}{k^2 \hbar^4} (1 - e^{-4k^2 r_0^2}) \tag{3}$$

HW:

Determine the scattering amplitude and the total cross section in case of a soft ball!

$$V(r) = \begin{cases} V_0, & \text{if } r < R \\ 0, & \text{if } r > R \end{cases}$$