# Addition of angular momentum

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# Revision of basic properties

- $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k, \quad [\mathbf{L}^2, L_i] = 0$
- $\mathbf{L}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle, \quad L_z|l,m\rangle = \hbar m|l,m\rangle$
  $l = 0, 1/2, 1, 3/2 \ldots \; m = -l, \ldots l,$
- $L_{\pm}|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l, m\pm 1\rangle$

# Addition of angular momentum

$$\mathbf{J} = \mathbf{L}_1 + \mathbf{L}_2, \qquad \mathbf{J}^2 = (\mathbf{L}_1 + \mathbf{L}_2)^2$$

Cartesian/Tensor product space: $\{|l_1, m_1\rangle |l_2, m_2\rangle\}$ of $(2l_1 + 1) \times (2l_2 + 1)$ dimension

$$J_z|l_1, m_1\rangle|l_2, m_2\rangle = (L_{1z} + L_{2z}|l_1, m_1\rangle|l_2, m_2\rangle = \hbar(m_1 + m_2)|l_1, m_1\rangle|l_1, m_1\rangle$$

The state $|l_1, m_1\rangle|l_2, m_2\rangle$ is an eigenstate of the operator $J_z = L_{1z} + L_{2z}$ with eigenvalue $m_1 + m_2$.

$$\mathbf{J}^2|l_1, m_1\rangle|l_2, m_2\rangle = (\mathbf{L}_1 + \mathbf{L}_2)^2|l_1, m_1\rangle|l_2, m_2\rangle \neq \alpha|l_1, m_1\rangle|l_2, m_2\rangle$$

The product $|l_1, m_1\rangle|l_2, m_2\rangle$ is not necessarily an eigenstate of $\mathbf{J}^2 = (\mathbf{L}_1 + \mathbf{L}_2)^2$, nevertheless its eigenstate can be constructed as linear combinations of the product states:

$$\mathbf{J}^2|j, m\rangle = (\mathbf{L}_1 + \mathbf{L}_2)^2 \sum_{m_1, m_2} c_{j,m;l_1,m_1,l_2,m_2}|l_1, m_1\rangle|l_2, m_2\rangle = \hbar^2 j(j+1)|j, m\rangle$$

Where the coefficients are called Clebsch-Gordan coefficients.

# Spin-1/2 (Spin one-half) particles

An He atom posseses of two spin one-half electrons. The Hamiltonian commutes with the operators $S_z = S_{1z} + S_{2z}$ and $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$.
Possible values for $S$: $1/2 - 1/2 = 0$ and $1/2 + 1/2 = 1$. There is only one way to construct the sates $|S, S_z\rangle = |1, 1\rangle$ and $|S, S_z\rangle = |1, -1\rangle$:

$$|1,1\rangle = |1/2,1/2\rangle|1/2,1/2\rangle \ , \qquad |1,-1\rangle = |1/2,-1/2\rangle|1/2,-1/2\rangle$$

Acting with the spin lowering operator $S_-$ on $|1,1\rangle = |1/2,1/2\rangle|1/2,1/2\rangle$:

$$S_-|1,1\rangle = (S_{1-} + S_{2-})|1/2,1/2\rangle|1/2,1/2\rangle$$

$$\sqrt{1(1+1) - 1(1-1)}|1,0\rangle = \sqrt{1/2(1/2+1) - (1/2(1/2-1)}|1/2,-1/2\rangle|1/2,1/2\rangle + \sqrt{1/2(1/2+1) - (1/2(1/2-1)}|1/2,1/2\rangle|1/2,-1/2\rangle$$

$$\sqrt{2}|1,0\rangle = |1/2,-1/2\rangle|1/2,1/2\rangle + |1/2,1/2\rangle|1/2,-1/2\rangle$$

From here:

$$|1,0\rangle = \frac{1}{\sqrt{2}}|1/2,-1/2\rangle|1/2,1/2\rangle + \frac{1}{\sqrt{2}}|1/2,1/2\rangle|1/2,-1/2\rangle$$

So both of the Clebsh-Gordan coefficients are $\frac{1}{\sqrt{2}}$.

# Singlet and triplet states

Two spin one-half particles can be in four different states:

**$S=1$ 3-fold degenerate triplet**

$$|1,1\rangle = |1/2,1/2\rangle|1/2,1/2\rangle$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}|1/2,-1/2\rangle|1/2,1/2\rangle + \frac{1}{\sqrt{2}}|1/2,1/2\rangle|1/2,-1/2\rangle$$
$$|1,-1\rangle = |1/2,-1/2\rangle|1/2,-1/2\rangle$$

**$S=0$ non-degenerate singlet**

$$|0,0\rangle = \frac{1}{\sqrt{2}}|1/2,-1/2\rangle|1/2,1/2\rangle - \frac{1}{\sqrt{2}}|1/2,1/2\rangle|1/2,-1/2\rangle$$

$|1,0\rangle$ and $|0,0\rangle$ contain the same states, but they must be orthogonal!

# $\mathbf{J}=\mathbf{L}+\mathbf{S}, \qquad l_1 = 1\,, \qquad l_2 = 1/2$

Possible values for $j$: $1-1/2=1/2$, $1+1/2=3/2$. States $|j_{max}, j_{max}\rangle$ and $|j_{max}, -j_{max}\rangle$ are always uniquely determined by the products: $|j_{max}, j_{max}\rangle = |l_1, l_1\rangle|l_2, l_2\rangle$ and $|j_{max}, -j_{max}\rangle = |l_1, -l_1\rangle|l_2, -l_2\rangle$. In our case

$$|3/2, 3/2\rangle = |1,1\rangle|1/2,1/2\rangle\,, \qquad |3/2,-3/2\rangle = |1,-1\rangle|1/2,-1/2\rangle$$

Similarly to what we have done before we act with the lowering ladder operators

$$J_-|3/2,3/2\rangle = (L_- + S_-)|1,1\rangle|1/2,1/2\rangle$$

$$\sqrt{3/2(3/2+1)-3/2(3/2-1)}|3/2,1/2\rangle = \sqrt{1(1+1)-1(1-1)}|1,0\rangle|1/2,1/2\rangle$$
$$+\sqrt{1/2(1/2+1)-1/2(1/2-1)}|1,1\rangle|1/2,-1/2\rangle$$
$$\sqrt{3}|3/2,1/2\rangle = \sqrt{2}|1,0\rangle|1/2,1/2\rangle + |1,1\rangle|1/2,-1/2\rangle$$

$$|3/2,1/2\rangle = \sqrt{\frac{2}{3}}|1,0\rangle|1/2,1/2\rangle + \frac{1}{\sqrt{3}}|1,1\rangle|1/2,-1/2\rangle$$

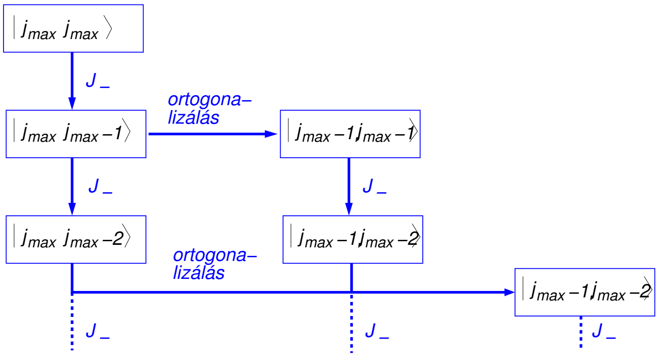
## $\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad l_1 = 1\ , \quad l_2 = 1/2$

$$|3/2, 1/2\rangle = \sqrt{\frac{2}{3}}|1,0\rangle|1/2,1/2\rangle + \frac{1}{\sqrt{3}}|1,1\rangle|1/2,-1/2\rangle$$

The state $|1/2, 1/2\rangle$ must contain the same products. Now make use of the orthogonality of the two states to get:

$$|1/2, 1/2\rangle = \frac{1}{\sqrt{3}}|1,0\rangle|1/2,1/2\rangle - \sqrt{\frac{2}{3}}|1,1\rangle|1/2,-1/2\rangle$$

# Algorithm


$|j_{max}\, j_{max}\rangle$
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lizálás
$|j_{max}-1\, j_{max}-2\rangle$
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# Exercises

1. Calculate the expectation value of $S_{2z}$ in the state $|1,1\rangle$ generated by two spin one-half particles, $\mathbf{S}_1$ and $\mathbf{S}_2$.
2. Determine the Clebsh-Gordan coefficients for the total angular momentum of a particle with $\mathbf{L}$, $l=1$ and $\mathbf{S}$, $s=1/2$.
3. Two particles with angular momentum $l=1$ move in spherical potential. The total angular momentum is conserved. What is the expectation value of the operator $L_{1z}$ in the state $|J,M\rangle = |1,1\rangle$?

## Homework

1. What is the total angular momentum of two particles with spin one-half and angular momentum $l$, respectively? Determine the Clebsh-Gordan coefficients! Use the binomial theorem to determine the powers of $J_- = L_- + S_-$! Show that for spin one-half particles $S_-{}^n = 0$, if $n > 1$ holds! Further hints: Then show that $(L_- + S_-)^n = L_-^n + nL_-^{n-1}S_-$. Act with $J_-^n$ on the state $|l+1/2, l+1/2\rangle$ and with $L_-^n + nL_-^{n-1}S_-$ on $|l,l\rangle\,|1/2,1/2\rangle$. For the latter it is worth first calculating the action of $L_-$ on $|l, l-k\rangle$, from which one can recursively find the coefficient for the action of $L_-^n$. Finally use the orthogonality relation for the coefficients of $|l-1/2, l-1/2-n\rangle$, $n = 0, 1, \ldots, 2l$.

# Solutions

1. Knowing from the solution of the first exercise, $|1,1\rangle = |1/2,1/2\rangle\,|1/2,1/2\rangle$ and that $S_{2z}$ acts only on the second part of the product state, $S_{2z}\,|1/2,1/2\rangle = \hbar/2\,|1/2,1/2\rangle$ we get:

$$\langle 1/2,1/2|S_{2z}|1/2,1/2\rangle = \hbar/2\,\langle 1/2,1/2|1/2,1/2\rangle = \hbar/2. \tag{1}$$

2. Find the solution in Quantum mechanical exercise collection, $4.10/a$

## Solutions

1. Find the solution of the decomposition of the state $|1,1\rangle$ in exercise $4.10/b$. Using this result, that is $|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle - \frac{1}{\sqrt{2}}|1,1\rangle|1,0\rangle$ and knowing that the $L_z$ operator only acts on the first states of the product state, $L_z|1,1\rangle|1,0\rangle = \hbar|1,1\rangle|1,0\rangle$, $L_z|1,0\rangle|1,1\rangle = 0$:

$$\langle 1,1|L_z|1,1\rangle = \left(\frac{1}{\sqrt{2}}\langle 1,0|\langle 1,1| - \frac{1}{\sqrt{2}}\langle 1,1|\langle 1,0|\right) L_z \left(\frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle - \frac{1}{\sqrt{2}}|1,1\rangle|1,0\rangle\right)$$

Only the terms with same quantum numbers survive, $\langle 1,0|1,1\rangle = 0$, as $L_z$ do not change these numbers:

$$\langle 1,1|L_z|1,1\rangle = \frac{1}{2}0\hbar + \frac{1}{2}\hbar = \frac{\hbar}{2}.$$