11. Practice, Dec. 10.

1. **Zitterbewegung**:
   Velocity operator in Heisenberg picture:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_k = \frac{i}{\hbar}\left[H, v_k\right] = \frac{ic}{\hbar}\left[c\alpha_l p_l + \beta mc^2, \alpha_k\right] = \frac{ic}{\hbar}\left(c_l\left[\alpha_l, \alpha_k\right] + mc^2\left[\beta, \alpha_k\right]\right)$$

where the emerging commutators:

$$\left[\alpha_l, \alpha_k\right] = \alpha_l\alpha_k - \alpha_k\alpha_l = \alpha_l\alpha_k + \alpha_k\alpha_l - 2\alpha_k\alpha_l = 2\delta_{kl}I_2 - 2\alpha_k\alpha_l$$

and

$$\left[\beta, \alpha_k\right] = \beta\alpha_k - \alpha_k\beta = \beta\alpha_k + \alpha_k\beta - 2\alpha_k\beta = -2\alpha_k\beta$$

Giving the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}v_k = \frac{ic}{\hbar}\,c(p_l\left(2\delta_{kl}I_2 - 2\alpha_k\alpha_l\right) - 2mc^2\alpha_k\beta) = \frac{2ic}{\hbar}\left(cp_k - \alpha_k\left(cp_l\alpha_l + mc^2\beta\right)\right) = \frac{2ic}{\hbar}\left(cp_k - \alpha_k H\right)$$

Considering the realtivistic momentum and energy

$$\mathbf{p} = \frac{m\mathbf{V}}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad E = \frac{mc^2}{\sqrt{1-\frac{V^2}{c^2}}}$$

yielding

$$\mathbf{V} = \frac{c^2\mathbf{p}}{E}$$

Analogously to this let us introduce the relativistic velocity operator:

$$\hat{V}_k = c^2\hat{p}_k\hat{H}^{-1}$$

being a constant of motion as it only contains $\hat{p}$ and $\hat{H}$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}v_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(v_k - V_k\right) = -\left(v_k - V_k\right)\frac{2i}{\hbar}H$$

From where the solution is

$$v_k(t) - V_k(t) = \left(v_k(0) - V_k(0)\right)e^{-i\frac{2H}{\hbar}t} \Rightarrow v_k(t) = V_k(t) + \left(v_k(0) - V_k(0)\right)e^{-i\Omega t}$$

where $\Omega \equiv \frac{2H}{\hbar}$ is the "frequency operator". From special relativity we would expect just $v_k = V_k$, but in quantum mechanics we have an extra oscillating term, that is the "Zitterbewegung" term. The coordiante operator then reads

$$x_k(t) = x_k(0) + V_k t + \left(v_k(0) - V_k(0)\right)i\Omega^{-1}\left(e^{-i\Omega t} - 1\right)$$

giving a motion along a straight trajectory with constant velocity and with an interpolating oscillatory motion.
Approximating the electron energy for small velocity as,$m_e c^2 \approx 511$ keV, gives $\Omega \sim 10^{21}$ Hz, unfortunately not measurable for the time being.
The amplitude of the motion is $\sim \frac{V}{\Omega} \sim \frac{\hbar}{mc} = \lambda_c \sim 10^{-13}$ m, being thousand times shorter than the radius of the hydrogen atom.