10. Practice, Nov. 29.

1. **Relativistic Landau niveaus**
The Dirac equation for a charged particle in uniform magnetic field can be written as

$$\begin{pmatrix} mc^2 - E & c\boldsymbol{\sigma}(\mathbf{p} - q\mathbf{A}) \\ c\boldsymbol{\sigma}(\mathbf{p} - q\mathbf{A}) & -(mc^2 + E) \end{pmatrix} \begin{pmatrix} \varphi_l \\ \varphi_s \end{pmatrix} = 0 \,,$$

where φ_l and φ_s are two-component spinors termed, which in the case of $E \geq mc^2$ are called the large and small components of the wavefunction, respectively, $\boldsymbol{\sigma}$ is the vector of the Pauli matrices. Let us consider the case. The small component can be expressed from the second line of Dirac equation,

$$\varphi_s = \frac{c\boldsymbol{\sigma}(\mathbf{p} - q\mathbf{A})}{mc^2 + E} \varphi_l \,,$$

and substituting it into the first line of the Dirac equation we get

$$(m^2c^4 - E^2 + c^2 \left(\boldsymbol{\sigma}(\mathbf{p} - q\mathbf{A})\right)^2)\varphi_l = 0 \ .$$

Let us examine the last term of the previous equation:

$$\begin{aligned}(\boldsymbol{\sigma}(\mathbf{p} - q\mathbf{A}))^2 &= \sigma_i\sigma_j(p_i - qA_i)(p_j - qA_j) = \delta_{ij}\mathbb{1}(p_i - qA_i)(p_j - qA_j) + i\epsilon_{ijk}\sigma_k(p_i - qA_i)(p_j - qA_j) \\ &= (\mathbf{p} - q\mathbf{A})^2 + i\epsilon_{ijk}\sigma_k \left(p_ip_j + q^2A_iA_j - q(p_iA_j + A_ip_j)\right) \,,\end{aligned}$$

$$i\epsilon_{ijk}\sigma_k \left((p_ip_j + q^2A_iA_j - q(p_iA_j + A_ip_j)\right) = i\epsilon_{ijk}\sigma_k \left(p_ip_j + q^2A_iA_j - q(A_jp_i + [p_i, A_j] + A_ip_j)\right)$$

Since ϵ_{ijk} is antisymmetric and $p_ip_j + q^2A_iA_j - q(A_jp_i + A_ip_j)$ is symmetric their product will disappear. The only term which survives the summation is

$$i\epsilon_{ijk}\sigma_k q[p_i, A_j] = \hbar q\epsilon_{ijk}\sigma_k \frac{\partial A_j}{\partial r_i} = -\hbar q\boldsymbol{\sigma}\mathrm{rot}\mathbf{A} = -\hbar q\boldsymbol{\sigma}\mathbf{B}$$

$$\Downarrow$$

$$(\boldsymbol{\sigma}(\mathbf{p} - q\mathbf{A}))^2 = (\mathbf{p} - q\mathbf{A})^2 - \hbar q\boldsymbol{\sigma}\mathbf{B} \,.$$

Therefore, the Dirac equation for the large component is given by

$$(m^2c^4 - E^2 + c^2(\mathbf{p} - q\mathbf{A})^2 - \hbar c^2 q\boldsymbol{\sigma}\mathbf{B})\varphi_l = 0 \,. \tag{1}$$

In case of $\mathbf{B}\|\mathbf{z}$, Tthe solutions can be searched in the form,

$$\varphi_{n+} = \varphi_n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \qquad \varphi_{n-} = \varphi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,.$$

where φ_n is eigenstate of the nonrelativistic Hamiltonian,

$$\frac{1}{2m} \left(\mathbf{p} - q\mathbf{A}\right)^2 \varphi_n = \varepsilon_n \varphi_n \ .$$

As we learned, the energy eigenvalues correspond to the Landau levels, $\varepsilon_n = \hbar\omega_L \left(n + \frac{1}{2}\right)$, where $\omega_L = \frac{|q|B}{m}$. The energy can be obtained from Eq. (1) as,

$$E_{n\pm} = \sqrt{m^2c^4 + 2mc^2\hbar\omega_L(n + \frac{1}{2}) \mp \hbar mc^2\omega_L} = mc^2\sqrt{1 + \frac{2\hbar\omega_L}{mc^2}\left(n + \frac{1}{2}\right) \mp \frac{\hbar\omega_L}{mc^2}} \,. \tag{2}$$

Expanding Eq. (2) to first order we obtain the well-known norelativistic limit,

$$E_{n\pm} = mc^2 + \hbar\omega_L \left(n + \frac{1}{2}\right) \mp \frac{\hbar\omega_L}{2} \,,$$

We further have to take care of the normalization of the bi-spinors,

$$\langle\psi_{n\sigma}|\psi_{n\sigma}\rangle = 1 + \langle\varphi_{n\sigma}|\frac{(c\boldsymbol{\sigma}(\mathbf{p}-q\mathbf{A}))^2}{(mc^2+E_{n\sigma})^2}|\varphi_{n\sigma}\rangle = 1 + \langle\varphi_{n\sigma}|c^2\frac{(\mathbf{p}-q\mathbf{A})^2 - \hbar q\boldsymbol{\sigma}\mathbf{B}}{(mc^2+E_{n\sigma})^2}|\varphi_{n\sigma}\rangle$$
$$= 1 + \frac{2mc^2\hbar\omega_L(n+\frac{1}{2}) - \sigma mc^2\hbar\omega_L}{(mc^2+E_{n\sigma})^2}, \ (\sigma = \pm)$$

So when φ_n is the eigenfunction of the Harmonic oscillator with energy $E_n = \hbar\omega_L\left(n+\frac{1}{2}\right)$, the properly normalized bi-spinor wavefunction is given by

$$|\psi_{n\sigma}\rangle = \begin{pmatrix} \varphi_l \\ \varphi_s \end{pmatrix} = \frac{1}{\sqrt{1 + \frac{2mc^2\hbar\omega_L(n+\frac{1}{2}) - \sigma mc^2\hbar\omega_L}{(mc^2+E_{n\sigma})^2}}} \begin{pmatrix} \varphi_{n\sigma} \\ \frac{c\boldsymbol{\sigma}(\mathbf{p}-q\mathbf{A})}{mc^2+E_{n\sigma}}\varphi_{n,\sigma} \end{pmatrix} . \tag{3}$$

2. **Spatial rotation and the total angular momentum operator**
As we have seen at the lecture class, the transformation of the wavefunction under a spatial rotation $R(\varphi, \vec{n})$ is given by

$$\psi'(R(\varphi,\vec{n})\vec{r}, t) = \exp\left(-\frac{i}{\hbar}\vec{n}\vec{S}\varphi\right)\psi(\vec{r},t) \ . \tag{4}$$

This can be rewritten in terms of the inversely rotated spacelike coordinates,

$$\psi'(\vec{r},t) = \exp\left(-\frac{i}{\hbar}\vec{n}\vec{S}\varphi\right)\psi(R(-\varphi,\vec{n})\vec{r},t) \ . \tag{5}$$

Let us express $\psi(R(-\varphi,\vec{n})\vec{r},t)$ for an infinitesimal rotation:

$$R(-\varphi,\vec{n})\vec{r} = (\vec{r}\cdot\vec{n})\vec{n} + (\vec{r} - (\vec{r}\cdot\vec{n})\vec{n})\cos\varphi - (\vec{n}\times\vec{r})\sin\varphi \approx \vec{r} - (\vec{n}\times\vec{r})\varphi$$

$$\Downarrow$$

$$\begin{aligned} \psi(R(-\varphi,\vec{n})\vec{r},t) &= \psi(\vec{r} - (\vec{n}\times\vec{r})\varphi, t) &(6)\\ &\simeq \psi(\vec{r},t) - \varphi\,(\vec{n}\times\vec{r})\,\vec{\nabla}\psi(\vec{r},t) \\ &= \psi(\vec{r},t) - \varphi\,\vec{n}\left(\vec{r}\times\vec{\nabla}\right)\psi(\vec{r},t) \\ &= \psi(\vec{r},t) - \frac{i}{\hbar}\varphi\,\vec{n}\,(\vec{r}\times\vec{p})\,\psi(\vec{r},t) \\ &= \left[I - \frac{i}{\hbar}\varphi\,\vec{n}\,\vec{L}\right]\psi(\vec{r},t) \ , &(7) \end{aligned}$$

consequently, for finite rotations,

$$\psi(R(-\varphi,\vec{n})\vec{r},t) = \exp\left(-\frac{i}{\hbar}\vec{n}\vec{L}\varphi\right)\psi(\vec{r},t) \ . \tag{8}$$

The complete transformation of the four component wavefunction under spatial rotations, $\psi(\vec{r},t) \rightarrow \psi'(\vec{r},t)$, can then be written as,

$$\begin{aligned} \psi'(\vec{r},t) &= \exp\left(-\frac{i}{\hbar}\vec{n}\vec{S}\varphi\right)\psi(R(-\varphi,\vec{n})\vec{r},t) = \exp\left(-\frac{i}{\hbar}\vec{n}\vec{S}\varphi\right)\exp\left(-\frac{i}{\hbar}\vec{n}\vec{L}\varphi\right)\psi(\vec{r},t) \\ &= \exp\left(-\frac{i}{\hbar}\vec{n}\vec{J}\varphi\right)\psi(\vec{r},t) \ , \end{aligned} \tag{9}$$

thus in relativistic quantum theory the infinitesimal generator of the spatial rotations is the *total angular momentum* operator,

$$\vec{J} = \vec{L} + \vec{S}\,.$$

3. **Klein paradox** Consider a free particle scattering on an infinitely wide potential well with height V_0:

$$V(z) = \begin{cases} 0, \text{ if } z < 0 \\ V_0, \text{ if } z > 0 \end{cases},$$

where for convenience we consider the potential step along the z direction. Let us look for the solution of the scattering problem for wave functions of the form,

$$\psi \sim e^{i\frac{pz}{\hbar}} \begin{pmatrix} 1 \\ 0 \\ \frac{pc}{E+mc^2} \\ 0 \end{pmatrix}.$$

Let us consider $E > mc^2$. Then for $z < 0$ the scattering solution of the one-dimensional Dirac equation,

$$\left(c\alpha_3 p_z + \beta mc^2\right)\psi = E\psi,$$

is

$$\psi(z) = Ae^{i\frac{pz}{\hbar}} \begin{pmatrix} 1 \\ 0 \\ \frac{pc}{E+mc^2} \\ 0 \end{pmatrix} + Be^{-i\frac{pz}{\hbar}} \begin{pmatrix} 1 \\ 0 \\ \frac{-pc}{E+mc^2} \\ 0 \end{pmatrix},$$

with $E^2 = p^2 + m^2c^4$. For $z > 0$,

$$z > 0 : \left(c\alpha_3 p_z + \beta mc^2\right)\psi = (E - V_0)\,\psi\,,$$

$$\psi(z) = Ce^{i\frac{p'z}{\hbar}} \begin{pmatrix} 1 \\ 0 \\ \frac{p'c}{E-V_0+mc^2} \\ 0 \end{pmatrix},$$

with $(E - V_0)^2 = (p')^2 + m^2c^4$. Note that for $E - mc^2 < V_0 < E + mc^2$ p' is purely imaginary. At $z = 0$ the continuity of the bi-spinor wavefunction yields

$$A + B = C$$

$$\frac{pc}{E+mc^2}A - \frac{pc}{E+mc^2}B = \frac{p'c}{E-V_0+mc^2}C$$

$$\Downarrow$$

$$A - B = -\frac{p'}{p}\frac{E+mc^2}{V_0 - E - mc^2}C \equiv -rC$$

$$\Downarrow$$

$$A = \frac{1-r}{2}C$$
$$B = \frac{1+r}{2}C$$

$$\Downarrow$$

$$\frac{B}{A} = \frac{1+r}{1-r}$$
$$\frac{C}{A} = \frac{2}{1-r} \, .$$

Exploiting the standard representation of $\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ we obtain for the incoming, reflected and transmitted current densities ($j = c\psi^{+}\alpha_3\psi$),

$$j_i = |A|^2 \frac{2pc^2}{E+mc^2}$$
$$j_r = |B|^2 \frac{-2pc^2}{E+mc^2}$$
$$j_t = |C|^2 \frac{2c^2 Imp'}{E - V_0 + mc^2} \, ,$$

respectively. From here one can easily get the reflection and transmission coefficients:

$$R = -\frac{j_r}{j_i} = -\frac{|B|^2}{|A|^2} = \frac{|1+r|^2}{|1-r|^2}$$
$$T = \frac{j_t}{j_i} = -\frac{|C|^2}{|A|^2} \frac{Imp'}{p} \frac{E+mc^2}{V_0 - E - mc^2} = -\frac{4 \mathrm{Im}\, r}{|1-r|^2} \, .$$

So we get

$$j_i + j_r = j_i \left(1 - \frac{|1+r|^2}{|1-r|^2}\right) = j_i \frac{-4 \mathrm{Im}\, r}{|1+r|^2} = j_t$$

or

$$R + T = \frac{|1+r|^2}{|1-r|^2} - \frac{4 \mathrm{Im}\, r}{|1-r|^2} = 1$$

as required by the continuity equation!
Now for determining the possible values of r, R and T we rewrite r as follows:

$$r = \frac{p'}{p} \frac{E+mc^2}{V_0 - E - mc^2} = \sqrt{\frac{(V_0 - E)^2 - m^2c^4}{E^2 - m^2c^4}} \frac{E+mc^2}{V_0 - E - mc^2} = \pm \sqrt{\frac{[V_0 - (E - mc^2)]\,(E+mc^2)}{[V_0 - (E+mc^2)]\,(E - mc^2)}}$$

We can distinguish between the following cases :

$$V_0 = 0 \Rightarrow r = -1, \; R = 0, \; T = 1, \text{ no scattering occurs}$$

$$V_0 > E - mc^2 \Rightarrow -1 < r < 0, \; 0 < R < 1, \; 0 < T < 1, \text{ normal scattering}$$

$$V_0 = E - mc^2 \Rightarrow r = 0, \; r = 0, \; R = 1, \; T = 0, \text{ total reflection}$$
$$E - mc^2 < V_0 < E + mc^2 \Rightarrow r = ir_0, \; r_0 > 0, \; R = 1, \; T = 0, \text{ total reflection}$$
$$V_0 = E + mc^2 \Rightarrow r = 0, \; r = 0, \; R = 1, \; T = 0, \text{ total reflection}$$

$$V_0 > E + mc^2 \Rightarrow r > \sqrt{\frac{E+mc^2}{E-mc^2}}, \; R > 1, \, T < 0, \text{ high potential step}$$
$$V_0 \to \infty \Rightarrow r = \frac{E+mc^2}{E-mc^2}, \; R = \left(\frac{\sqrt{E+mc^2} + \sqrt{E-mc^2}}{\sqrt{E+mc^2} - \sqrt{E-mc^2}}\right), \; T = -4\frac{\sqrt{(E+mc^2)(E-mc^2)}}{\left(\sqrt{E+mc^2} - \sqrt{E-mc^2}\right)}$$

For $E - mc^2 < V_0 < E + mc^2$ the wavefunction is exponentially decaying for $z > 0$, thus there is no transmitted current density. However, for $V_0 > E + mc^2$ we get $R > 1$ and $T < 0$. This can be interperted as the potential creates electron-positron (particle-antiparticle) pairs: the excess reflection corresponds to electrons travelling back in the $-z$ direction, while $T < 0$ corresponds to positrons going to the $+z$ direction.

4. **Homework: Spatial reflection**