# A simple statement about the period of periodic motions

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Let us take a one dimensional Hamiltonian, for simplicity we assume that it is time independent:

$$H(x,p) \tag{0.1}$$

We assume that the solutions are closed orbits $x(t)$ such that

$$x(t+T) = x(t), \quad \text{where} \quad T = T(E) \tag{0.2}$$

We introduce the action angle variables $(\varphi, I)$. Here

$$2\pi I = \oint dq\ p \tag{0.3}$$

The parameter $I$ is time independent, because it is integrated over the full orbit. Therefore it only depends on the overall energy, which characterizes the orbits:

$$I = I(E) \tag{0.4}$$

This relation can be understood also in the other direction: the energy is constant along the path, therefore after the canonical transformation $(x,p) \to (\varphi, I)$ the Hamiltonian only depends on the action variable:

$$H(\varphi, I) = H(I) \tag{0.5}$$

The Hamilton equations are:

$$\dot{I} = 0, \qquad \dot{\varphi} = \frac{dH}{dI} \tag{0.6}$$

The right hand side is a constant in time, therefore $\varphi(t) = \frac{dH}{dI}t + c$.

What is the change of $\varphi$ during one period? In the lecture you learned that this is exactly $2\pi$ by definition. It follows that

$$T = \frac{2\pi}{\frac{dH}{dI}} = 2\pi \frac{dI}{dE} \tag{0.7}$$

Here we used again that the energy and thus $H$ only depends on $I$ and not on the angle. This is what we wanted to prove.

**Proof of the statement that $\varphi$ is changed by $2\pi$ over one period**

Here we just provide this proof once again.

Let consider $\varphi$ as a function of the action and the original position: $\varphi(I,q)$. We compute the change as the contour integral (integral back and forth)

$$\Delta\varphi = \oint d\varphi = \oint \frac{\partial \varphi}{\partial q} dq \tag{0.8}$$

The $\varphi, I$ pair is a canonical pair, which means that

$$\frac{\partial\varphi(I,q)}{\partial q} = \frac{\partial p(I,q)}{\partial I} \tag{0.9}$$

This can be obtained from a generating function, or directly from the canonical transformation. Now we compute it directly from the canonical transformation.

The $(\varphi, I)$ pair is canonical if

$$\{\varphi, I\} = \left.\frac{\partial\varphi}{\partial q}\right|_p \left.\frac{\partial I}{\partial p}\right|_q - \left.\frac{\partial I}{\partial q}\right|_p \left.\frac{\partial\varphi}{\partial p}\right|_q = 1 \tag{0.10}$$

Here both $\varphi$ and $I$ are understood as functions of $(q,p)$, therefore we denoted the variable which is kept fixed in the subscript. The difference with (0.9) is that in the Poisson bracket the variables are the original $(q,p)$, and we have to change to the mixed parametrization $(q,I)$.

Let us consider changing $p, q$ such that $I$ is kept constant. Then

$$\left.\frac{\partial p}{\partial q}\right|_I = -\frac{\left.\frac{\partial I}{\partial q}\right|_p}{\left.\frac{\partial I}{\partial p}\right|_q} \tag{0.11}$$

Substituting into (0.10)

$$\left.\frac{\partial\varphi}{\partial q}\right|_p \left.\frac{\partial I}{\partial p}\right|_q + \left.\frac{\partial p}{\partial q}\right|_I \left.\frac{\partial I}{\partial p}\right|_q \left.\frac{\partial\varphi}{\partial p}\right|_q = 1 \tag{0.12}$$

which gives

$$\left.\frac{\partial I}{\partial p}\right|_q \left[\left.\frac{\partial\varphi}{\partial q}\right|_p + \left.\frac{\partial p}{\partial q}\right|_I \left.\frac{\partial\varphi}{\partial p}\right|_q\right] = 1 \tag{0.13}$$

or

$$\left.\frac{\partial p}{\partial I}\right|_q = \left(\left.\frac{\partial I}{\partial p}\right|_q\right)^{-1} = \left.\frac{\partial\varphi}{\partial q}\right|_p + \left.\frac{\partial p}{\partial q}\right|_I \left.\frac{\partial\varphi}{\partial p}\right|_q = \left.\frac{\partial\varphi}{\partial q}\right|_I \tag{0.14}$$

and this is what we wanted to prove, see eq. (0.9). In any case, this relation is easier found using a generating function, when

$$\left.\frac{\partial\varphi}{\partial q}\right|_I = \frac{\partial^2 W(q,I)}{\partial q \partial I} = \left.\frac{\partial p}{\partial I}\right|_q \tag{0.15}$$

Returning to (0.8) we get

$$\Delta\varphi = \oint \left.\frac{\partial p}{\partial I}\right|_q dq = \frac{\partial}{\partial I}\oint p dq = 2\pi\frac{\partial I}{\partial I} = 2\pi \tag{0.16}$$