Problem 4

A particle of resting mass m, and electric charge q is in a static homogeneous electric and magnetic fields E and B that are perpendicular to each other. The initial velocity of the particle is zero. Determine the motion of the particle. The magnetic inducion points in the z direction while the electric field points in the y direction.

1. Write down the relativistic equations of motion for the particle in the covariant form (like in Problem 4).
2. We could solve simply the equations of a.) (as a practice you can do it.), but now it's worth to follow a different way. Our argument is the following: in a crossed electric and magnetic field one can figure out a uniform linear motion, where the magnetic Lorentz-force and the electric force cancel each other. If we boost to a frame that moves with the velocity of that motion, the electric field strength must be zero, because our particle is in rest in that frame. Here we can solve the much easier problem, where only a magnetic field is present, and finally we transform back to the original frame of reference, and get the solution of our problem.
3. What is the velocity of the uniform linear motion? When is it physically meaningful?
4. Transform the field-strength tensor to that frame!
5. Solve the problem in the moving frame!
6. Transform back to the original frame, and express $x(t)$. Sketch the trajectory of the particle.
7. What happens if the velocity in b.) is not physically meaningful? How looks like the trajectory of the particle in that case?

Solution:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{28}$$

The invariant quantities are

$$FF = F_{\mu\nu}F^{\mu\nu} = \vec{B}^2 - \vec{E}^2 \qquad FG = F^{\mu\nu}F^{\delta\kappa}\varepsilon_{\mu\nu\delta\kappa} = 2\vec{E}\cdot\vec{B} \tag{29}$$

In our case we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E_y & 0 \\ 0 & 0 & -B_z & 0 \\ E_y & B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

The condition for the vanishing force is

$$E_y - B_z v_x = 0 \tag{31}$$

The minus sign comes from the fact that we used covariant vector here!

This gives

$$v_x = \frac{E_y}{B_z} \tag{32}$$

Clearly it is meaningful only when $FF > 0$.

The Lorentz transformation we want is

$$\Lambda = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) & 0 & 0 \\ -\sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

with $\tanh(\theta) = v_x$.

As matrices

$$F' = \Lambda F \Lambda^T \tag{34}$$

It can be checked that the vector

$$\begin{pmatrix} E_y \\ B_z \end{pmatrix} \tag{35}$$

transforms as a two-vector under the 2x2 transformation

$$\Lambda = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix} \tag{36}$$

In the new coordinates the initial condition is

$$x'^\mu(0) = 0, \qquad v'_x(0) = -v, \quad v_y(0) = 0, \quad v_z(0) = 0 \tag{37}$$

What is the new magnetic field? It is given by

$$B_z'^2 = \vec{B}^2 - \vec{E}^2 = B_z^2 - E_y^2 \tag{38}$$

We copy from earlier:

$$\omega = \frac{qB'_z\sqrt{1 - v^2/c^2}}{m} \qquad r = \frac{v}{\omega} = \frac{mv}{qB'_z\sqrt{1 - v^2/c^2}} \tag{39}$$

Here we can actually use

$$1 - v^2/c^2 = 1 - \frac{E_y^2}{B_z^2} = \frac{B_z^2 - E_y^2}{B_z^2} = \frac{(B'_z)^2}{B_z^2} \tag{40}$$

So it is better to put

$$\omega = \frac{q(B'_z)^2}{B_z m} \qquad r = \frac{v}{\omega} = \frac{mvB_z}{q(B'_z)^2} = \frac{mE_y}{q(B'_z)^2} \tag{41}$$

Here the solution with the proper initial conditions will be

$$x'(t) = -r\sin(\omega t') \qquad y(t') = r(1 - \cos(\omega t')) \tag{42}$$

So the 4-vector is

$$x'^\mu(t') = \begin{pmatrix} t' \\ -r\sin(\omega t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix} \tag{43}$$

Now we transform back with

$$\Lambda = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

We can use

$$\cosh(\theta) = \frac{B_z}{B'_z} \qquad \sinh(\theta) = \frac{E_y}{B'_z} \tag{45}$$

This gives

$$\begin{pmatrix} \frac{1}{B'_z}(B_z t' - E_y r\sin(\omega t')) \\ \frac{1}{B'_z}(-B_z r\sin(\omega t') + E_y t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{B'_z}(B_z t' - \frac{mE_y^2}{q(B'_z)^2}\sin(\omega t')) \\ \frac{E_y}{B'_z}(-\frac{mB_z}{q(B'_z)^2}\sin(\omega t') + t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{B'_z}(B_z t' - \frac{mE_y^2}{q(B'_z)^2}\sin(\omega t')) \\ \frac{E_y}{B'_z}(-\frac{\sin(\omega t')}{\omega} + t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix} \tag{46}$$

We could also write this as

$$t = \frac{1}{B'_z}\left(B_z t' - \frac{mE_y^2}{q(B'_z)^2}\sin(\omega t')\right) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{mE_y}{q(B'_z)^2}\begin{pmatrix} \frac{B_z}{B'_z}(-\sin(\omega t') + \omega t') \\ (1-\cos(\omega t')) \end{pmatrix} \tag{47}$$

It is useful to check the small time limit, when everything is non-relativistic. Expandint the function $t(t')$ to first order in t' we obtain

$$t = t'\frac{B'_z}{B_z} \tag{48}$$

Also, $x(t)$ starts with t^3, so we will not investigate this. Instead we will look at $y(t)$, which is of order t^2 for small t. Expanding the solution we get

$$y(t) \approx \frac{mE_y}{q(B'_z)^2}\omega^2\frac{t'^2}{2} = \frac{mE_y}{q(B'_z)^2}\left(\frac{q(B'_z)^2}{B_z m}\right)^2\left(\frac{B_z^2}{(B'_z)^2}\frac{t^2}{2}\right) = \frac{qE_y}{2m}t^2 \tag{49}$$

This is the expected result!

The case of $|B_z| < |E_y|$

In the case when B'_z becomes imaginary, so would t', and we have different functions. We put

$$t' = i\tilde{t} \qquad B'_z = iE'_y = i\sqrt{E_y^2 - B_y^2} \tag{50}$$

and

$$t = \frac{1}{E'_y}\left(B_z\tilde{t} + \frac{mE_y^2}{q(E'_y)^2}\sinh(\omega\tilde{t})\right) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{mE_y}{q(E'_y)^2}\begin{pmatrix} \frac{B_z}{E'_y}(-\sinh(\omega\tilde{t}) + \omega\tilde{t}) \\ (1 - \cosh(\omega\tilde{t})) \end{pmatrix} \tag{51}$$

In the large $\tilde{t}$ limit

$$t \approx \frac{mE_y^2}{q(E'_y)^3}\frac{e^{\omega\tilde{t}}}{2} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{mE_y}{q(E'_y)^2}\frac{e^{\omega\tilde{t}}}{2}\begin{pmatrix} \frac{B_z}{E'_y} \\ 1 \end{pmatrix} \tag{52}$$

We can read off that the speed becomes 1, as it should be, and the direction depends on the magnetic and electric fields.