Problem 1

We return to the Runge-Lenz vector and the Kepler problem.

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} \tag{1}$$

(a) Consider the following vector (it is called the Runge-Lenz vector)

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\alpha \mathbf{e}_r \tag{2}$$

Express the components of that vector so, that only the components of the canonical momentum and position vectors are used.

(b) Determine the Poisson bracket $[A_j, H]$. What does it tell about the vector $\mathbf{A}$?

(c) Using the conservation of $\mathbf{A}$, derive the particle's orbital.

To do so consider an orbital that is in the x-y plane and the vector A points in the x direction. (We can always choose a coordinate system where this is true.) Express the scalar product $\mathbf{A} \cdot \mathbf{r}$ using polar coordinates and then using trivial algebra express the orbital.

(d) What does the direction of A mean? What is its length?

Problem 2

A particle of mass m can move in the $x - y$ plane where a conservative $V(x, y)$ potential is also present.

(a) Write down the Lagrangian of the system and determine the Hamiltonian as a function of p_x, p_y, x and y.

(b) We would like to transform to the rotating frame. The transformation is described by

$$\begin{aligned} x(t) &= X(t)\cos(\omega t) - Y(t)\sin(\omega t) \\ y(t) &= X(t)\sin(\omega t) + Y(t)\cos(\omega t) \end{aligned} \tag{3}$$

Write down the Lagrangian in the X, Y variables.

(c) Determine the "new" Hamiltonian K as a function of X,Y,P_X and P_Y. What is the connection between the "new" and the "old" Hamiltonian?

(d) Show that the Poisson brackets between the variables x, y, p_x, p_y don't change if we calculate them using the new canonical coordinates X, Y, P_X, P_Y.

Problem 3

The Hamiltonian of a two-dimensional isotropic harmonic oscillator reads as

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) \tag{4}$$

(a) Show that the elements of the following 2x2 matrix are conserved quantities:

$$A_{ij} = \frac{1}{2}\left(\frac{1}{m}p_i p_j + m\omega^2 x_i x_j\right) \tag{5}$$

(b) What kind of symmetry is generated by the elements of the matrix A?

(c) Introduce the following quantites:

$$S_1 = \frac{A_{12}}{\omega} \qquad S_2 = \frac{A_{22} - A_{11}}{2\omega} \qquad S_3 = \frac{L}{2} = \frac{1}{2}(xp_y - yp_x) \tag{6}$$

Show that their Poisson brackets are $\{S_i, S_j\} = \varepsilon_{ijk} S_k$.

(d) Show that $H^2 = 4\omega^2(S_1^2 + S_2^2 + S_3^2)$! (we did not treat this in class)