Problem 1

A (nonrelativistic) particle of mass m and charge e moves in a homogeneous magnetic field B that points in the z direction. The gravitational and electric fields are zero. The magnetic vectorpotential is chosen to be $A = (0, Bx, 0)$. (This is called the Landau-gauge)

(a) Write down the Lagrangian of the system.

(b) Show that the Lagrangian is invariant under the transformation $x' = x$, $y' = y$, $z' = z + s$. Using Noether's theorem express the corrseponding conserved quantity.

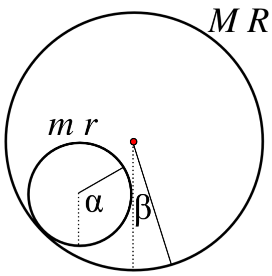
(c) Show that the transformation $x' = x$, $y' = y + s$, $z' = z$ is also a symmetry. What is the corresponding conserved quantity?

(d) Consider the transformation $x' = x + s$, $y' = y$, $z = z$. This Lagrangian is not invariant under this transformation. Show that the change in the Lagrangian is a total time derivative in this case. We know that the equations of motion are invariant, if we add such a term to the Lagrangian: the transformation is a symmetry of the problem.

(e) Construct the corresponding conserved quantity of problem

Problem 2

An empty cilinder of mass M and radius R can easily rotate around its symmetry axis. Within the cilinder there is a smaller ring of mass m and radius r. The configuration of the system is described by the angles α and β. The gravitational field is described by the vertical vector g.



(a) Construct the Lagrangian of the system. Show that it reads as

$$L = mr^2\dot{\alpha}^2 + \frac{1}{2}(m+M)R^2\dot{\beta}^2 - mrR\dot{\alpha}\dot{\beta} + mg(R-r)\cos\left(\frac{r\alpha - R\beta}{R-r}\right) \quad (1)$$

(b) Show that the transformation $\alpha' = \alpha + \phi$, $\beta' = \beta + \phi\frac{r}{R}$ keeps the Lagrangian invariant. What kind of ,,motion" is described by this transformation?

(c) Using Noether's theorem, determine the corresponding conserved quantity.

(d) Determine the equations of motion for the system. Show directly, that the conserved quantity derived in c.) is indeed a constant of motion.

Problem 3

Consider a particle in a central potential. The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + V(r) \quad (2)$$

(a) Write down the components of the angular momentum (L_x, L_y and L_z) using the canonical momentum $\mathbf{p}$ and the position $\mathbf{x}$.

(b) Determine the Poisson brackets $[L_x, x]$, $[L_x, y]$, $[L_x, p_x]$ and $[L_x, p_y]$.

(c) Generalize the results of b.), so determine the Poisson brackets $[L_i, r_j]$ and $[L_i, p_j]$ for any i, j indices.

(d) Determine the Poisson bracktes $[L_i, L_j]$.

(e) Determine the Poisson brackets $[L_i, H]$ for any value of i. What does this tell about the angular momentum?

Problem 4

Consider the Kepler problem that is described by the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} \tag{3}$$

(a) Consider the following vector (it is called the Runge-Lenz vector)

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\alpha \mathbf{e}_r \tag{4}$$

Express the components of that vector so, that only the components of the canonical momentum and position vectors are used.

(b) Determine the Poisson bracket $[A_j, H]$. What does it tell about the vector $\mathbf{A}$?

(c) Using the conservation of $\mathbf{A}$, derive the particle's orbital.

To do so consider an orbital that is in the x-y plane and the vector A points in the x direction. (We can always choose a coordinate system where this is true.) Express the scalar product $\mathbf{A} \cdot \mathbf{r}$ using polar coordinates and then using trivial algebra express the orbital.

(d) What does the direction of A mean? What is its length?