Problem 1

Consider the transversal waves of an elastic rod. The cross-section parameter of the rod is Θ, its linear mass density is $A\rho$, and its Young's modulus is E. The Lagrangian of the system reads as

$$\mathcal{L} = \frac{1}{2}\rho A(\dot{u})^2 - \frac{\Theta E}{2}(u'')^2 \tag{1}$$

The two ends are fixed horizontally in two walls, therefore at the ends both the displacement and its z-derivative is zero.

(a) Write down the action for the system.

(b) Using the principle of least action derive the equations of motion for the system.

(c) Search the solution in the separated form: $u(z,t) = U(z)\varphi(t)$. Write down the appropriate equations for $U(z)$ and $\varphi(t)$.

(d) Write down the equation that determines the free oscillation frequencies of the system. Qualitatively solve the equation, graphically.

Problem 2

One of the most important non-quadratic field theories is the sine-Gordon model for a field $\varphi(x,t)$, described by the Lagrangian

$$L = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 + \cos(\phi) - 1 \tag{2}$$

(a) Derive the Euler-Lagrange equations of motion for the model.

(b) Determine the expression of the energy density in the model.

(c) Derive the expression of the energy current in the model.

(d) Search for constant (in time and space) solutions that solve the equations of motion. What is the energy density in these solutions? Which configurations of these have finite total energies?

(e) We would like to find such solutions that transfer from one of these configurations to the other. Show that the following time-independent configuration solves the equations

$$\varphi_1(x,t) = 4\arctan(e^x) \tag{3}$$

Note.: This solution is called a standing soliton. Hint:

$$\sin(4\arctan(y)) = \frac{4(y-y^3)}{(1+y^2)^2} \tag{4}$$

(f) Write down the function in the $x \to \pm\infty$. Sketch the function.

(g) Determine the energy density, and its integral (the total energy) for this solution.

(h) Show that the following time-dependent solution solves the equations.

$$\varphi_2(x,t) = 4\arctan\left(e^{\frac{x-vt}{\sqrt{1-v^2}}}\right) \tag{5}$$

Note.: This is called the moving soliton solution with velocity v.

(i) Determine the energy density, end its integral (the total energy) for the solution φ_2. Hint:

$$\cos(4\arctan(y)) = 1 - 8\frac{y^2}{(1+y^2)^2} \tag{6}$$

Determine the energy current for the solution φ_2. Sketch it as a function of x.