Problem 1

(repeating from the last lecture)

Problem 2

Consider the longitudinal waves traveling in a thin, elastic rod. The longitudinal displacement of the points of the rod is described by the field $\xi(x,t)$. The Young's modulus of the rod is E, its mass density is ρ, and its cross-secion is A.

(a) Write down the (linear-) density of the kinetic energy as a function of the time-derivative of the field.

(b) Write down the (linear-) density of the elastic energy as a function of the x-derivative of the field.

(c) Write down the Lagrangian of the system.

(d) Using the principle of least action determine the equations of motion for the system.

(e) From the action, determine the expression for the (total) energy density in the system.

(f) Write down the energy of in a finite piece of the system. Determine its time-derivative.

(g) Determine the expression for the energy-current in the system. Derive the continuity equation for the energy.

Problem 3

Consider the transversal waves of an elastic rod. The cross-section parameter of the rod is Θ, its linear mass density is $A\rho$, and its Young's modulus is E. The Lagrangian of the system reads as

$$\mathcal{L} = \frac{1}{2}\rho A(\dot{u})^2 - \frac{\Theta E}{2}(u'')^2 \tag{1}$$

The two ends are fixed horizontally in two walls, therefore at the ends both the displacement and its z-derivative is zero.

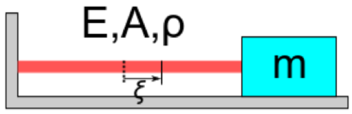
(a) Write down the action for the system.

(b) Using the principle of least action derive the equations of motion for the system.

(c) Search the solution in the separated form: $u(z,t) = U(z)\varphi(t)$. Write down the appropriate equations for $U(z)$ and $\varphi(t)$.

(d) Write down the equation that determines the free oscillation frequencies of the system. Qualitatively solve the equation, graphically.

Problem 4



A body of mass m is fixed to the end of an elastic rod. The cross-section of the rod is A, its Young's modulus is E, its mass-density is ρ, and the length of the rod is L.

The longitudinal displacement of the points of the rod is described by the field $\xi(z,t)$, the transversal displacement of the rod is negligible in our case. The position of the body is described by $u(t)$. The action of the system is

$$S = \int dt \left\{ \frac{1}{2} m (\dot{u})^2 + A \int dx \left[\frac{\rho}{2} (\dot{\xi})^2 - \frac{E}{2} (\xi')^2 \right] \right\} \tag{2}$$

As we see, if one naiively derives the equations of motion of this action, one gets a trivially wrong result: the body will not be fixed to the end of the rod. We have to include the constraint $\xi(L,t) = u(t)$ explicitely in the calculations.

(a) The constraint can be taken into account using a (time-dependent) Lagrange-multiplicator. Write down the action modified by the Lagrange multiplicator.

(b) Write down the variation of the action.

(c) Following the usual way, using integrations by part transform the action in a form where only the variations are present, while their derivatives are not. In the case of $\delta\xi$, be careful at the boundary $z = L$.

(d) Using the principle of least action derive the equations of motion for the system.

(e) Search for the solutions in the following wave-form:

$$\xi(x,t) = A \sin(\omega t) \sin(kx) \tag{3}$$

Determine the connection between k and ω.

(f) Starting from the solution of e.) write down the equation of motion for the body. You should arrive to a transcendent equation for the possible k values. Don't solve the equation.

(g) Discuss the limit, when the mass of the body is negligible. What are the free oscillation frequencies of the system in that case?

(h) Discuss the limit, when the mass of the rod is negligible. What is the smallest free oscillation frequency in that case?

(i) Determine the energy density and energy current in the rod. Show that the energy current that "flows out" from the rod at the body is exactly the time derivative of the body's kinetic enegy.