Problem 1

Consider the one-parameter subgroup of Lorentz transformations that containes the boosts in the x direction. In that case one can simply forget the y and z coodinates because these are not transformed. Consequently it is sufficient to consider only the top left 2×2 block of the Lorentz matrix. In the lecture it was shown that in this special case, the Lorentz matrix can be parametrized as

$$\Lambda(\theta) = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix} \tag{1}$$

(a) What is the connection between the parameter θ (the rapidity) and the velocity v of the boost?

(b) Show that the above transformation has the following property

$$\Lambda(\theta_1)\Lambda(\theta_2) = \Lambda(\theta_1 + \theta_2) \tag{2}$$

(c) By the use of this property, derive the "rule of addition" for relativistic velocities. What is the meaning of this formula?

(d) Two relativisticly fast cars are traveling by $0.8c$ towards each other. According to one of the drivers, what is the velocity of the other car?

Problem 2

In the lecture the 4-velocity vector $u^\mu = \frac{dx^\mu}{d\tau}$ has been introduced, and it has been shown that this is a proper 4-vector.

(a) Write down the connection between the 4-velocity and the usual (3-)velocity vector.

(b) Let's suppose, that watching the sky, we see two spacecrafts that are flying towards each other, and both have velocity 0.6c. We use a coordinate system, where the trajectories of the spacecrafts lie on the x-axis.

Determine the 4-velocities of the spacecrafts.

(c) Write down a Lorentz-transformation that transforms into the frame of one of the spacecrafts.

(d) Express the 4-velocities of the spacecrafts in that frame of reference.

(e) What is the usual 3-velocity of the spacecrafts in that frame? Let suppose now, that – as we see from the Earth – the two spacecrafts travel in perpendicular directions, x and y.

(f) Determine the modified 4-velocities of the spacecrafts

(g) Transform to the frame of the spacecraft travelling in the x direction. What is the 4-velocity of the other spacecraft in this frame?

(h) What is the usual 3-velocity of the other spacecraft in this frame?

Problem 3

The Compton effect (Artur Holly Compton 1892 – 1962. Nobel-prize: 1927) was one of the important experimental results that led to the birth of quantum mechanics. This experiment showed that a photon of energy $\hbar\omega$ has also a momentum $\hbar\omega/c$. Here ω is the frequency of the photon.

In the experiment a photon of frequency ω_0 collides with an initially resting electron (mass m). After the collision the electron has a momentum p while the photon loses from its energy, and its trajectory distorts by an angle of ϑ. After the collision we detect the scattered photon.

(a) Define a convenient coordinate system. Sketch a figure about the process.

(b) Write down the total 4-momentum of the system before and after the collision.

(c) Determine the frequency ω' of the scattered photon as a function of the distortion angle ϑ. Exploit the conservation of 4-momentum.

Problem 4

Let's consider the elastic collision of two particles. The particles move on a common, straight trajectory. One has resting mass m_1 and (usual) velocity v_1 while the other has resting mass m_2 and velocity v_2. Their common trajectory defines the x-axis.

(a) Write down the 4-momenta p_1^μ and p_2^μ of the two particles. What is the meaning of their components?

(b) Write down the equation for the 4-momentum conservation. It's scary, isn't it?

(c) In non-relativistic collision problems it is a neat trick to transform of the frame of the "center of mass". In this frame, the 4-momentum conservation gives a much simpler equation, and one can immediately write down the momenta after the collision. Let's try to generalize this trick for the relativistic case.

(d) Write down the total 4-momentum of the system before the collision.

(e) Write down the matrix of a Lorentz boost with some arbitrary velocity V, and transform the 4-momentum with this transformation.

(f) What should be V, if we want the total (3-)momentum to vanish in the moving frame? Let's define this velocity as the velocity of the "center of mass".

(g) Transform to the frame of the center of mass. Express the 4-momenta of the particles in that frame before and after the collision.

(h) Transform back to the original frame, and express the 4-momenta of the particles after the collision.