Problem 1

Consider a two-dimensional anisotropic oscillator. The Hamiltonian of the system is

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2) \tag{1}$$

(a) Write down the full (time dependent) Hamilton-Jacobi equation for the system.

(b) The Hamiltonian does not depend on time, therefore the Hamilton-Jacobi equation can be separated in the form $S(x, y, t) = S_0(x, y, E) - Et$. Write down the abbreviated Hamilton-Jacobi equation for S_0.

(c) Separate further the function S_0, i.e. look for the solution in the form

$$S_0(x, y, E) = S_x(x, \alpha_x) + S_y(y, \alpha_y) \tag{2}$$

Write down the equations for S_x and S_y. Denote the new constants by $\alpha_{x,y}$.

(d) Determine the functions S_x, S_y, and express the full solution $S(x, y, \alpha_x, \alpha_y, t)$ of the Hamilton-Jacobi equation.

(e) The particle is initially ($t = 0$) at the position $x = x_0$ and $y = y_0$, and has zero momentum. Determine the values of the constants.

(f) How can one get the $x(t)$, $y(t)$ solutions of the equations of motion, using $S(x, y, t)$? (Don't calculate it! It's a lengthy calculation.)

Problem 2

Two identical particles of mass m are connected by a spring whose spring-constant is D. The particles can move along the x axis. The Hamiltonian of the system is

$$H(x_1, p_1, x_2, p_2) = \frac{p_1^2 + p_2^2}{2m} + \frac{D}{2}(x_1 - x_2)^2 \tag{3}$$

(a) Write down the full (time dependent) Hamilton-Jacobi equation for the system.

(b) The Hamiltonian does not depend on time, therefore the Hamilton-Jacobi equation can be separated int the form $S = S_0 - Et$. Write down the abbreviated Hamilton-Jacobi equation for S_0.

(c) Further separation cannot be done using the coordinates x_1 ,x_2. Transform the equation to the new variables $X = (x_1 + x_2)/2$ and $y = x_1 - x_2$ and rewrite the equation of b.) using these variables.

(d) Separate the S_0 function as $S_0(x_1, x_2, E) = S_y(y, \alpha_y) + S_X(X, \alpha_X)$. Write down the equations for S_X and S_y! Denote the new constants by α_y, α_X.

(e) Determine the functions S_X and S_y.

(f) Knowing the initial conditions $(x_{1,0}, x_{2,0}, p_{1,0}, p_{2,0})$ determine the values of the $\alpha_{X,y}$ parameters.

Problem 3

Two identical bodies can move along the x axis in a box. The two bodies are attached to the walls through two springs with spring constant D, and there is also a spring between the two bodies. The Hamiltonian of the system is

$$H(x_1, p_1, x_2, p_2) = \frac{p_1^2 + p_2^2}{2m} + \frac{D}{2}(x_1^2 + x_2^2) + \frac{D}{2}(x_1 - x_2)^2 \tag{4}$$

(a) Write down the abbreviated Hamilton-Jacobi equation for the system.

(b) The equation is not separable immediately, using the variables x_1 and x_2. Transform to the new variables $X = (x_1 + x_2)/2$ and $y = x_1 - x_2$. Show that the equation is now separable. Perform the separation.

(c) Solve the H.J. equation using the separation.

(d) Determine the oscillation frequencies in the system. Is the motion of the system periodic for any initial conditions?

Problem 4

Consider the following generalized oscillator, that is described by a power-law potential with exponent $\alpha > 0$:

$$H(p, x) = \frac{p^2}{2m} + k|x|^\alpha \tag{5}$$

(a) Draw the contour lines $H(p, x) = E$ on the $p - x$ plane.

(b) Determine the integral that equals the phase-surface bounded by the contour-lines. Denote it by $2\pi I$.

(c) In the generic case the integral cannot be analytically determined. The best we can do is to determine the (power-law) dependence on the parameters E, m, and k. Performing appropriate variable transformations make the integral dimensionless, i.e. collect all the dependence on the parameters outside the integral. In this case the value of the dimensionless integral is only a number, that can be calculated numerically.

(d) Using the derivation of $I(E)$ determine the period of the oscillation as a function of the parameters.

Problem 5

The "adiabatic inveriance" of the action variable is an interesting theorem of Hamiltonian mechanics. The theorem, whose proof can be found in [1,2]), states that in a system with one degree of freedom the value of the action variables remains constant, even if we slowly change the parameters of the system, i.e. the Hamiltonian is (slightly) time-dependent. The understanding of the theorem is easier, if we introduce a time-dependent parameter in the Hamiltonian, i.e.

$$H = H(x, p, \lambda(t)) \tag{6}$$

The action variable at a given value of λ can be determined uby calculating the phase-surface bounded by the equienergetic contour:

$$2\pi I = \oint p(E, q, \lambda) dq \tag{7}$$

The theorem states, that if λ varies slowly and smoothly, then the value of I remains constant.

Consider a pendulum, where we slowly shrink the length of the pendulum. The Hamiltonian of the system for small amplitudes is

$$H = \frac{p_\varphi^2}{2ml^2(t)} + \frac{1}{2} mgl(t)\varphi^2 \tag{8}$$

(a) For a given length l determine the action variable $I(l, E)$.

(b) We start from a small initial (angular) amplitude A, when the length of the pendulum is l_0. Then we slowly shrink the length of the pendulum to $l_0/2$. What is the final angular amplitude of the pendulum?

[1] H. Goldstein, Classical Mechanics [2] Clive G Wells and Stephen T C Siklos, Eur. J. Phys. 28, 105 (ArXiV:physics/0610084)