Problem 1

A two-dimensional, charged harmonic oscillator is put into homogeneous magnetic field. The field vector is perpendicular to the plane of the oscillator. The Hamiltonian of the system (using the so called symmetric gauge) reads as

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{2}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{eB}{2}x\right)^2 + \frac{1}{2}m\omega^2(x^2+y^2) \tag{1}$$

We solve the problem using a strange but elegant trick. The main idea relies on the observation, that the form of the magnetic Lorentz force and the Coriolis force in rotating frames is very similar. Therefore, if one transforms in an appropriate rotating frame, the effect of the magnetic field will be compensated by the Coriolis force. As a result, we get a simple harmonic oscillator, that we can solve easily.

(a) Consider the following generator function, and show that it transforms to the rotating frame.

$$W_2(x, y, P_X, P_Y, t) = x(P_x\cos(\Omega t) - P_Y\sin(\Omega t)) + y(P_X\sin(\Omega t) + P_Y\cos(\Omega t)) \tag{2}$$

Using the derivatives of the generator function, express the variables p_x, p_y, X and Y as functions of x,y,P_X,P_Y.

(b) Using the relations of a.) express the "old" variables $\{x, y, p_x, p_y\}$ in terms of the new ones $\{X, Y, P_X, P_Y\}$.

(c) Determine the new form of the Hamiltonian $K(X, Y, P_X, P_Y)$. If your calculations are correct, it is time independent.

(d) Show that the new Hamiltonian can be transformed to the form of the original one, i.e.

$$K = \frac{1}{2m}\left(P_x + \frac{eB'}{2}Y\right)^2 + \frac{1}{2m}\left(P_y - \frac{eB'}{2}X\right)^2 + \frac{1}{2}m\omega'^2(X^2+Y^2) \tag{3}$$

where B' and ω' are the new effective magnetic field and oscillator frequencies, respectively. Express these in terms of the Ω angular velocity and B, e, m and ω.

(e) What should be Ω if we want the effective magnetic field B' to vanish? What is the form of the Hamiltonian K in that case, i.e. what is the oscillator frequency ω'?

(f) Solve the equations of motion in the system, where $B' = 0$. We know, that the particle moves on elliptic orbitals. What is the time period of the motion?

(g) If we transform back to the original (inertial-) frame, the axis of the ellipse will rotate by angular velocity Ω. Sketch the trajectory of the particle.

Problem 2

Consider the problem of the vertical motion in a homogeneous gravitational field. The Hamiltonian of the system is

$$H(p, x) = \frac{p^2}{2m} + mgx \tag{4}$$

- Write down the full (time dependent) Hamilton-Jacobi equation for the system.
- Because the Hamiltonian does not depend on time explicitly, we can look for the function $S(x, t)$ in the form $S(x, t) = S_0(x, E) - Et$, where E is a constant. Express the abbreviated Hamilton-Jacobi equation for the function S_0.
- Solve the equation for S_0.
- Knowing the function $S(x, E, t)$, determine the canonical transformation that it generates. Express the canonical coordinate β_E, that is the canonical pair of E.
- The particle is initially in the position $x = 0$ and has momentum p_0. Using this information determine the values of E and β_E.
- Express the $x(t)$ solution of the equation of motion.

Problem 3

Consider a one-dimensional harmonic oscillator, whose Lagrangian is

$$L = \frac{1}{2}mv^2 - \frac{1}{2}m\omega^2 x^2 \tag{5}$$

(a) Write down and solve the Lagrangian equations of motion.

(b) Find a solution with boundary conditions $x(0) = 0$ and $x(t_f) = x_f$!

(c) Express the action S for the solution of b). Denote it by $S(x_f, t_f)$!

(d) Show that the function S solves the Hamilton-Jacobi equation of the system!

(e) What is $\frac{\partial S}{\partial t_f}$?

Problem 4

The Schrödinger equation of a particle moving in the one-dimensional potential $V(x)$ is the following

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\Psi(x,t) + V(x)\Psi(x,t) \tag{6}$$

It is maybe useful to look for the solution in the form

$$\Psi(x,t) = A(x,t)e^{\frac{i}{\hbar}S(x,t)} \tag{7}$$

(a) Substitute this form of Ψ in the Schrödinger equation.

(b) Collect the terms according to the powers of $\hbar$.

(c) If is small enough then it is sufficient to use the zeroth order term only. What equation you get in that case?

(d) When is $\hbar$ "small enough"?