Problem 1

Let η be a 2 component vector constructed from a pair of canonical coordinates $\{q, p\}$. Consider the following transformation

$$\xi = \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \log(\sin(p)/q) \\ q\cot(p) \end{pmatrix} \tag{1}$$

(a) Compute the Jacobi matrix $M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}$.

(b) Show that the transformation is canonical, i.e. it keeps the simplectic structure J_{ij} invariant:

$$J_{ij} = M_{ik} J_{kl} M_{jl} \tag{2}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

Problem 2

Consider the following generator functions

$$\begin{aligned} W_1(q, Q) &= qQ \\ W_2(q, P) &= qP \\ W_3(p, Q) &= pQ \\ W_4(p, P) &= pP \end{aligned} \tag{4}$$

(a) Determine the generated canonical transformations for each generator function.

(b) By checking the poisson brackets of Q and P show that the transformations are canonical.

Problem 3

Consider a linear harmonic oscillator whose Hamiltonian reads

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{5}$$

Consider the following generator functions and try to derive transformation rules.

(a) $W_1(q, Q) = q + Q$

(b) $W_2(q, P) = q + P$

(c) $W_2(q, P) = (q + P)^2$

(d) Which generator function describes indeed a transformation? Perform the transformation and determine the ,,new" Hamiltonian $K(Q, P)$.

(e) Determine the canonical equations using the new form of the Hamiltonian. Solve the equations!

Problem 4

In the lecture you studied the following (2nd type) generator function

$$W_2(q, P) = \sum_l f_l(q) P_l \tag{6}$$

that leads to the "point-transformation"

$$Q_l = f_l, \qquad p_l = \frac{\partial f_m}{\partial q_l} P_m \tag{7}$$

(a) Consider the transformation from Descartes- to polar coordinates:

$$r = \sqrt{x^2 + y^2} \qquad \phi = \arctan(x/y) \tag{8}$$

Determine the generator function for this transformation.

(b) Express the "old" momenta p_x and p_y in terms of P_r and P_ϕ.

(c) Invert the expressions, and express the "new" momenta P_r and P_ϕ in terms of the "old" momenta.

Problem 5

Consider a linear harmonic oscillator whose Hamiltonian reads as

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{9}$$

(a) Write down the canonical equations, solve them, and draw the trajectories in the phase-space $p-q$.

(b) We see that because of the conservation of energy the trajectories are ellipses in the pase space. It seems to be a good idea to make these ellipses to coordinate lines. This can be done using the parametrization

$$p = m\omega A\cos(Q) \qquad q = A\sin(Q) \tag{10}$$

We will see that the parameter A has to be some function of P, which we don't know yet.

Search for a 1st type generator function that leads to this transformation.

(c) Determine the new momentum and coordinate in terms of the old variables.

(d) Express the new form of the Hamiltonian, determine the canonical equations and solve them.

Problem 6

The Hamiltonian of a system with one degree of freedom reads as

$$H = \frac{1}{2m}(p^2 e^{-2\alpha t} + m^2\omega^2 q^2 e^{2\alpha t}) \tag{11}$$

(a) Determine the canonical equations. What kind of system is this?

(b) Explain that the canonical momentum "p" is not equal to the physical momentum "mv". What is the connection between them?

(c) Consider the following generator function

$$W_1(q, Q) = -qQe^{\alpha t} - \frac{\alpha m}{2} q^2 e^{2\alpha t} \tag{12}$$

Determine the generated canonical transformation.

(d) Express the new form K(Q,P) of the Hamiltonian. Be careful! The generator function depends explicitly on time.

(e) Determine the equations of motion in the new variables and solve them.

(f) Transform back the solutions to the original p, x coordinates.