A1

The Lagrangian of a two-dimensional isotropic harmonic oscillator reads as

$$L(x, \dot{x}, y, \dot{y}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m\omega^2}{2}(x^2 + y^2) \tag{1}$$

Consider the following transformation:

$$\begin{aligned} x' &= x\cos(\phi) + y\sin(\phi) \\ y' &= -x\sin(\phi) + y\cos(\phi) \end{aligned} \tag{2}$$

(a) How do the velocities transform?

(b) Show that the Lagrangian is invariant under the transformation.

(c) Using Noether's theorem, derive the corresponding conserved quantity.

(d) Write down the equations of motion for the system.

(e) Show directly, that the quantity derived in c.) is indeed a constant of motion.

A2

The Hamiltonian of a system with two degrees of freedom reads

$$H = q_1p_1 - q_2p_2 - Aq_1^2 + Bq_2^2, \tag{3}$$

where A and B are real parameters.

(a) Write down the equations of motion (Hamilton's canonical equations) for the system..

(b) Consider the following quantities:

$$F_1 = \frac{p_1 - Aq_1}{q_2} \qquad F_2 = q_1q_2 \tag{4}$$

Calculate the Poisson brackets $[F_1, H]$ and $[F_2, H]$. Show that both quantities are constants of motion.

(c) Calculate the Poisson bracket $[F_1, F_2]$. Let $F_3 = [F_1, F_2]$. Is it a constant of motion?

B1

A wire track is fixed to a cylinder (see Figure). A small body of mass m can move without friction on the track The moment of inertia of the cilinder is Θ, and it can rotate around its axis. The position of the system is described by the rotation angle ϕ of the cilinder and the position α of the body. The gravitational force acts on the body. The coordinates of the small body are descrbied by

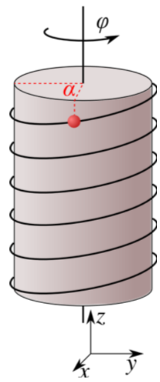
$$x = R\sin(\alpha) \qquad y = -R\cos(\alpha) \qquad z = C(\alpha - \phi) \tag{5}$$

where C is the vertical slope of the track.

- Construct the Lagrangian of the system.
- Show that the following transformation is a symmetry:

$$\alpha' = \alpha + c \qquad \phi' = \phi + c \tag{6}$$

- Using Noether's theorem, derive the corresponding conserved quantity.
- Determine the equations of motion of the system.
- At the moment $t = 0$ the system starts from the position $\alpha = \phi = 0$, with zero velocities. Solve the equations of motion.
- Determine the value of the quantity in c.) and show that it is indeed a constant of motion.



B2

Consider the Problem 2.) of class (see Figure).

(a) Using Legendre transformation determine the Hamiltonian of the system.

(b) Express the conserved quantity, that has been derived in class, using the canonical momentum and coordinates. Denote it by A.

(c) Calculate the Poisson bracket $[A, H]$, and show that the quantity is indeed conserved.