Important: If something is not clear, please ask your questions during the Consultation!

A1

Consider a very long elastic rod with mass density ρ, Young's modulus E, and cross-section A. The rod lies on the x axis. One end of the rod is in the origin ($x = 0$), the other end is far away, practically in the infinity ($x \to \infty$. The longitudinal waves in the rod are described by the Lagrangian density

$$L = \frac{\rho A}{2}(\partial_t \xi(x,t))^2 - \frac{EA}{2}(\partial_x \xi(x,t))^2, \tag{1}$$

where the field $\xi(x,t)$ describes the longitudinal displacement of the points of the rod.

(a) Write down the Euler-Lagrange equation of motion for the system.

(b) We excite the rod by moving its end ($x = 0$) as $\xi(0,t) = a\sin(\omega t)$. Show directly that following plane-wave solution solves the equations of motion: $\xi(x,t) = a\sin(\omega t - kx)$. Determine the value of k as a function of ω.

(c) Starting from the Lagrangian, derive the expression for the energy density ε, and the energy current J_ε. (Use the law of energy conservation, similarly to the problems in class)

(d) Determine the average power of the generator that excites the rod. (Hint: calculate the total transmitted energy in one period $T = 2\pi/\omega$)

A2

Consider the bending waves in an elastic rod. As it has been discussed in class, the bending energy is proportional to the square of the curvature of the rod, therefore the Lagrangian of the system is

$$L = \frac{\lambda}{2}(\partial_t u(x,t))^2 - \frac{E\Theta}{2}(\partial_x^2 u(x,t))^2, \tag{2}$$

where λ is the linear mass density (mass per unit length), E is the Young's modulus of the rod, and Θ is the cross-section parameter. The field $u(x,t)$ describes the small transversal displacement of the rod.

(a) Determine the Euler-Lagrange equation of motion for the system. Be careful with the second-derivative term.

(b) Starting from the Lagrangian, determine the energy density in the model.

(c) Determine the expression of energy current in the model. (Try to follow the way of the class: calculate the time-derivative of the energy density, and use the equations of motion to transform it into a total x-derivative. In this case the energy current is a sum of two different terms.)

B1

We consider an uniaxial ferromagnetic spinchain, and instead of treating the full quantum problem we build an effective classical field theoretical description. In this model the energy density is

$$\varepsilon = \frac{1}{2}\partial_x \mathbf{M} \cdot \partial_x \mathbf{M} - \frac{\lambda}{2} M_z^2, \tag{3}$$

where the first term describes that the neighboring spins want to be parralel to each other, while the second term shows, that the spins prefer the $\pm z$ directions. The length of the spins is fixed, therefore we can fix the length of the $\mathbf{M}$ vectorfield to $\mathbf{M} \cdot \mathbf{M} = 1$. This constraint could be taken into account by using a Lagrange-multiplicator but here we follow a different way.

(a) Parametrize the vector field using spherical coordinates:

$$\mathbf{M} = \begin{pmatrix} \sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) \\ \cos(\theta) \end{pmatrix} \tag{4}$$

Write down the energy density using the new fields θ and ϕ.

(b) By varying the total energy, determine the equations determining the equilibrium configurations.

(c) Search for constant solutions. How many do you find? What is their stability?

(d) Search for such (smallest energy) solution, that transfers from one constant solution to an other constant solution (domain wall).