# A1

In this problem you will show that the basic equation of electrostatics (the Poisson-equation) can be deduced using a variational principle. Consider the stationary (time-independent) charge density $\rho(r)$, whose electrostatic energy is described by the following functional

$$U = \int d^3r \left(-\frac{\varepsilon_0}{2}(\nabla\phi)^2 + \rho\phi\right) \tag{1}$$

where $\phi(r)$ denotes the electrostatic potential from which the electric field can be determined by $E = -\nabla\phi$.

The variational principle states that the electricstatic field minimizes the electrostatic energy. The boundary conditions usually are described by $\phi(|r| \to \infty) = 0$, i.e. the potential is zero in the infinity.

Assuming that we change the potential by the infinitezimal variation $\delta\phi(r)$ write down the variation $\delta U$ of the energy functional.

After integrating by parts transform the variation $\delta U$ in a form

$$\delta U = \int d^3r M(r) d\phi(r) \tag{2}$$

i.e. the derivatives of the variation $d\phi(r)$ are not present anymore. Determine $M(r)$ as a function of $\rho(r)$ and $\phi(r)$. Throw out the boundary terms of the partial integration. (It can be shown that if $\rho(r)$ is confined in a finite volume then the boundary terms are exactly zero.)

The energy is minimal, if $\delta U$ is zero for any variation. It is equivalent to the equation $M(r) = 0$. Show that this equation is exactly the Poisson-equation of electrostatics.

# B1

The Lagrangian of a homogeneous, elastic, but not isotropic medium is described by

$$\mathcal{L} = \frac{\rho}{2}(\dot{\mathbf{s}})^2 - \frac{\lambda}{2}(\nabla\mathbf{s})^2 - \mu\sum_j(\partial_j s_j)^2 \tag{3}$$

where $\mathbf{s}(\mathbf{x}, t)$ is the field that describes the displacement of the medium, the mass density is $\rho$, $\lambda$ and $\mu$ are the Lamé parameters of the medium.

(a)

(b) Write down the action of the system.

(c) Derive the equations of motion for the system.

(d) Show that the following plane-wave expression solves the equations

$$\mathbf{s} = \mathbf{s_0} e^{i(\mathbf{kx} - \omega t)} \tag{4}$$

(e) By inserting the solution of c) in the equations of motion, show that the connection between $\mathbf{s}_0$, $k$ and $\omega$ reads as

$$M(\omega, k)\mathbf{s}_0 = 0 \tag{5}$$

Determine the matrix $M$.

(f) Let the wave propagate in the $x$ direction. Determine the matrix $M$ in this special case.

(g) The equation in d.) can only be solved if the determinant of the matrix $M$ is zero. Using the special form in e.), determine the possible $\omega(k)$ dispersion relations.

(h) Determine the $\mathbf{s}_0$ amplitude vectors for the different $\omega(k)$ solutions. What kind of polarizations appear?

# B2

Consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2m}\partial_x \Psi^* \partial_x \Psi - V(x)\Psi^*\Psi + \frac{1}{2}i(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*), \tag{6}$$

where $\Psi(x,t)$ is a complex valued field, and $\Psi^*(x,t)$ denotes its complex conjugate. There are many ways to handle complex fields. Now we follow the most pedestrian way: we describe the field as a combination of two independent real fields.

(a) Consider the complex field as a real field with two-components (the real and the imaginary part.) Here $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are standard real fields. Rewrite the Lagrangian in the terms of these two real fields.

(b) Show that the Lagrangian is real (no complex factors are present).

(c) Write down the action using the real form of the Lagrangian.

(d) Derive the equations of motion for the two fields $\Psi_{1,2}$.

(e) Show that the two equations are the real and imaginary parts of the usual Schrödinger equation.

# B3

A soap-membrane is streched on a square shaped frame of size $L$. In this problem we consider the standing wave modes of the membrane. The surface mass density of the membrane is $\lambda$, the surface tension is $\sigma$. The Lagrangian of the membrane reads as

$$\mathcal{L} = \frac{1}{2}\lambda(\partial_t u)^2 - 2\sigma\sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2}, \tag{7}$$

where $u(x,y,t)$ is the vertical displacement of the membrane. The first term is the surface density of kinetic energy, while the second term stands for the surface energy density of the membrane.

(a) Approximate the square-root term in the Lagrangian up to orders of $(\partial_{x,y}u)^2$

(b) Derive the equations of motion (from the approximated Lagrangian).

Because of the boundary conditions, at the frame the displacement is zero, $u(0,y,t) = u(x,0,t) = u(L,y,t) = u(x,L,t) = 0$.

(c) Look for the standing-wave solutions in the form

$$u = A\sin(k_x x)\sin(k_y y)\sin(\omega t) \tag{8}$$

What are the possible values of $k_x$ and $k_y$ that are allowed by the boundary conditions?

(d) Determine the frequency $\omega$ as a function of the wave numbers $k_x$ and $k_y$.