# Relativistic mechanics

Galilei: • frame of reference

origin $\quad O$

3 directions $\quad (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ } space

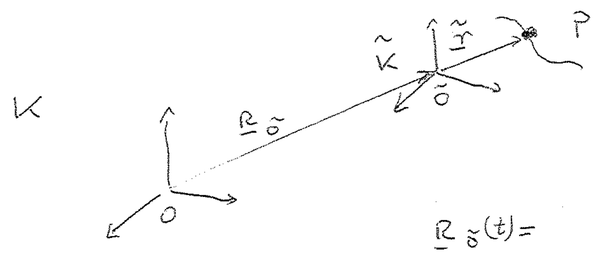
+ time $\quad t$

• ∃ inertial frame of reference

in these frames objects are at rest or follow straight lines unless force acts on them

- inertial frames move along straight lines w.r.t. each other

- transformation of coordinates (Galilei)




$$\underline{R}_{\tilde{O}}(t) = \underline{R}_0 + \underline{W} \cdot t \quad , \qquad (*)$$

$$\tilde{\underline{r}}(t) + \underline{R}_{\tilde{O}}(t) = \underline{r}(t)$$

⇒ velocity transformation / addition

$$\boxed{\underline{v} = \tilde{\underline{v}} + \underline{W}} \qquad (*)'$$

• Relativity:

no way to tell which frame (inertial) moves ...

Maxwell: in vacuum $\boxed{\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\underline{E} = \phi}$ $\quad (**)$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

light moves with velocity $c$ ...

contradiction:

<u>Case A</u>: (*) is correct $\Rightarrow$ (**) can work in a single frame of refrene $\equiv$ ETHER

a single inertial frame of referhee

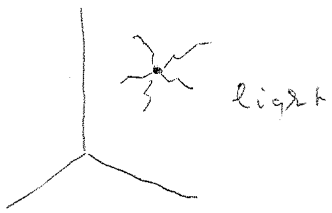
<u>Case B</u>: (**) is correct $\Rightarrow$ (*) is wrong...

<u>Tests</u>: • Michelson – Morley (1887)

$\Rightarrow$ no ether

$\boxed{\text{light propagates in any inertial frame of reference with velocity } c \text{, along straight line}}$

Lorentz – Einstein




$$d\underline{x}^2 = c^2 dt^2 \qquad d\underline{\tilde{x}}^2 = c^2 d\tilde{t}^2$$

$$\Rightarrow \boxed{\underbrace{c^2 dt^2 - d\underline{x}^2 \equiv ds^2}_{\text{invariant !}}}$$

Generalize Galilei transformation

"event" = "now and here" = $\{x^\mu\} \equiv (ct, \underline{x}) \equiv X$

four vector $\qquad \mu = 0, 1, 2, 3$

$\Rightarrow \tilde{X} \equiv \{\tilde{x}^\mu\} = (c\tilde{t}, \underline{\tilde{x}}) = ?$

• map must be linear! (image of straight line is straight line!)

$$\tilde{x}^\mu = \sum_\nu L^\mu{}_\nu x^\nu + a^\mu \qquad \boxed{\tilde{X} = A + L \cdot X}$$

Lorentz transformation $\qquad$ Poincaré introduce

$$G \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \equiv \{g^{\mu\nu}\} \equiv \{g_{\mu\nu}\}$$

<u>metric</u> <u>tensor</u>

$$ds^2 = dX^T G\, dX \overset{!}{=} d\tilde{X}^T G\, d\tilde{X} = dX^T L^T G L\, dX$$

$$\boxed{G \overset{!}{=} L^T G L} \quad \longleftrightarrow \quad \text{orthogonal matrices } \mathbb{1} = O^T O$$

defines a group : $O(3,1)$ or $O(1,3)$

<u>Lorentz group</u> : $L \in \mathcal{L}$

$L_1$ and $L_2 \in \mathcal{L} \Rightarrow L_1 L_2 \in \mathcal{L}$

$L_1^{-1}, L_2^{-1} \in \mathcal{L}$

<u>take 2×2</u> <u>$O(1,1)$</u> <u>case</u> :

$$L_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad g_2 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma^2 & \alpha\beta - \gamma\delta \\ \cdot\cdot & \beta^2 - \delta^2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\alpha \equiv \mathrm{ch}\,\varphi \cdot \sigma_t$ ; $\sigma_t \equiv \pm 1$ $\qquad \delta \equiv \sigma_x \cdot \mathrm{ch}\,\vartheta$

$\gamma \equiv -\mathrm{sh}\,\varphi$ $\qquad \beta \equiv -\mathrm{sh}\,\vartheta$

case $\sigma_t = \sigma_x = 1 \Rightarrow \vartheta = \varphi$

$$L_2(\varphi) = \begin{pmatrix} \mathrm{ch}\varphi & -\mathrm{sh}\varphi \\ -\mathrm{sh}\varphi & \mathrm{ch}\varphi \end{pmatrix}$$

~ rotation in hyperbolic space!

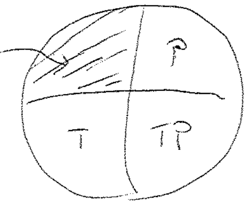
parity : $P \equiv \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ ; $T \equiv \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$

time reversal

$$\mathcal{L}_2 = \{ L_2(\varphi),\ P L_2(\varphi),\ T L_2(\varphi),\ PT\, L_2(\varphi) \}$$

ordinary L-transformations ~ subgroup

$\mathcal{L}_2$



identify $\varphi$

(1) $\quad c\tilde{t} = ct\,\mathrm{ch}(\varphi) - x\,\mathrm{sh}(\varphi)$

(2) $\quad \tilde{x} = -\mathrm{sh}(\varphi)\cdot ct + x\cdot \mathrm{ch}(\varphi)$

origin $\tilde{O}$ : $\tilde{x} = 0 \Rightarrow x = \underbrace{c\cdot \mathrm{th}(\varphi)}_{w} \cdot t$ !

$$\mathrm{ch}^2(\varphi) = \frac{1}{1-\mathrm{th}^2(\varphi)} = \frac{1}{1-w^2/c^2} \Rightarrow \mathrm{sh}^2(\varphi) = \frac{w^2/c^2}{1-w^2/c^2}$$

$$\Rightarrow \boxed{\tilde{t} = \frac{t - \frac{w}{c^2}x}{\sqrt{1-w^2/c^2}}}, \quad \boxed{\tilde{x} = \frac{x - w\cdot t}{\sqrt{1-w^2/c^2}}}$$

Inverse: $\quad L_2(\varphi_1)\cdot L_2(\varphi_2) = L_2(\varphi_1+\varphi_2) \Rightarrow$

$$(L_2(\varphi))^{-1} = L_2(-\varphi) \Rightarrow w \leftrightarrow -w \quad !$$

velocity addition:

invert (1+2) $(\varphi \to -\varphi)$

$$dt = d\tilde{t}\cdot C + \frac{d\tilde{x}}{c}\cdot S$$

$$dx = +S\cdot c\,d\tilde{t} + C\cdot d\tilde{x}$$

$$\Rightarrow \boxed{v_p = \frac{dx}{dt} = \frac{S\cdot c\,d\tilde{t} + C\,d\tilde{x}}{C\,d\tilde{t} + \frac{d\tilde{x}}{c}\cdot S} = \frac{w + \tilde{v}_p}{1 + \frac{\tilde{v}_p\cdot w}{c^2}}}$$

- $c \to \infty$ ✓
- $\tilde{v}_p = c$ (light) $\to v_p = c$ ✓

Physical consequences:

- time dilation:

"proper time" in $\tilde{K}$ $\qquad d\tilde{x} = 0 \qquad d\tau = d\tilde{t}$

start clock and wait first tick $\to d\tau$

$\Rightarrow$ in $K$: $\quad dt = ch(\varphi) \cdot d\tilde{t} + \emptyset = \dfrac{d\tau}{\sqrt{1 - w^2/c^2}}$

$\quad dx = \emptyset + sh(\varphi) \cdot c \cdot d\tilde{t}$

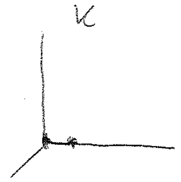
$dt > d\tau$ !

- tests:
  - $\mu$-on short lived particle (lepton) $\to$ detected on earth
  - supersonic airplanes + atomic clocks!
  - Alvager et al (CERN) 1964 $\quad \pi^0 \to \gamma\gamma$, $w \approx c$; $\gamma\gamma$ move with $c$

  $\vdots$

- Lorentz-contraction

distance ~ equal time measurement,



$\Rightarrow$ ends of standing rod in $\tilde{K}$

$\tilde{x}_1 = 0 \qquad \tilde{x}_2 = \tilde{L}$

moves in $K$!

"orbit"

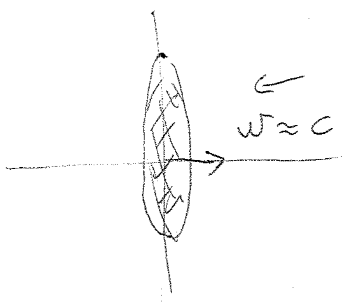
$\tilde{x}_1 = 0 = \dfrac{x_1 - w \cdot t_1}{\sqrt{1 - w^2/c^2}}$, $\tilde{x}_2 = \tilde{L} = \ldots$

measure distance at given time $t_1 \equiv t_2 \equiv \emptyset$, e.g.

$\Rightarrow \quad x_1 = 0 \qquad x_2 = \sqrt{1 - w^2/c^2} \cdot \tilde{x}_2$

$\Rightarrow \quad L = x_2 - x_1 = \sqrt{1 - w^2/c^2} \cdot \tilde{L}$

length of moving object squeezed!



$\leftarrow$ $w \approx c$

moving ball, as we observe it in case we take a picture...

Lecture 2

## General case:

Lorentz group:

rotations:

$$R_z(\vartheta) = \begin{pmatrix} 1 & & & \\ & \cos\vartheta & -\sin\vartheta & \\ & \sin\vartheta & \cos\vartheta & \\ & & & 1 \end{pmatrix}, R_x, R_y$$

boosts:

$$K_x(\varphi) = \begin{pmatrix} \mathrm{ch}\varphi & -\mathrm{sh}\varphi & & \\ -\mathrm{sh}\varphi & \mathrm{ch}\varphi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad K_y, K_z$$

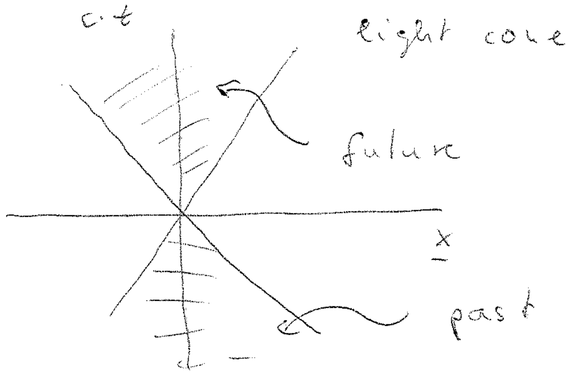
Remarks:
- $K_x(\varphi_x)\, K_y(\varphi_y) \neq K_y(\varphi_y)\, K_x(\varphi_x)$

  $R_x(\vartheta_x)\, K_z(\varphi_z) \neq K_z(\varphi_z)\, R_x(\vartheta_x)$ etc.
- proper L-group $\det L = 1$, $L_{00} \geq 1$

  P and T $\Rightarrow$ full L-group
- Symmetry of space-time includes translations

  $\Rightarrow$ Poincaré

## Minkowski space:




time-like separation
two events have a causal relation if

$\Delta x^\mu_{AB} = x^\mu_A - x^\mu_B$ time-like

$\sim ds^2_{AB} > 0$ !

invariant under $\mathcal{L}$

Four - vectors : covariant and contravariant representations

Def: $A$ is a four-vector if $\boxed{\tilde{a}^{\mu} = L^{\mu}{}_{\nu}\, a^{\nu}}$

summation

Def: $Q$ is a four-tensor if

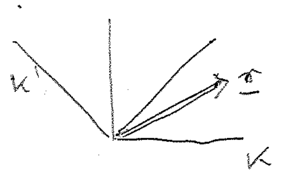
$$\boxed{\tilde{Q}^{\mu\nu} = L^{\mu}{}_{\mu'} L^{\nu}{}_{\nu'} Q^{\mu'\nu'}}$$

in general: $\tilde{Q}^{\mu\nu\ldots\rho} = L^{\mu}{}_{\mu'} L^{\nu}{}_{\nu'} \ldots L^{\rho}{}_{\rho'}\, Q^{\mu'\nu'\ldots\rho'}$

physicist's definition!

• think of rotations!

$\underline{r}$




$\underline{r}$ → $(x^1, x^2, x^3)$ in $K$

→ $(\tilde{x}^1, \ldots \tilde{x}^3)$ in $\tilde{K}$

$\tilde{\underline{x}} = \underline{\underline{R}}\, \underline{x}$

orthogonal matrix

stress tensor $\hat{\sigma} \Rightarrow \sigma^{ij} = \underline{\underline{\sigma}}$ in $K$

$\tilde{\sigma}^{ij} = \tilde{\underline{\underline{\sigma}}}$ in $K'$

$$\tilde{\sigma}^{ij} = R^{i}{}_{i'}\, \sigma^{i'j'}\, R^{j}{}_{j'}$$

$$\tilde{\underline{\underline{\sigma}}} = \underline{\underline{R}}\, \underline{\underline{\sigma}}\, \underline{\underline{R}}^{T}$$

$A$ has two representations:

contravariant : $\{a^{\mu}\} = \{a^0, \underline{a}\}$

covariant : $\{a_{\mu}\} \equiv \{a^0, -\underline{a}\}$

$a_{\mu} = g_{\mu\nu} a^{\nu}$

Def: scalar product:

$$\boxed{A \cdot B \equiv a^{\mu} b_{\mu}} = a^{\mu} g_{\mu\nu} b^{\nu} =$$

$$= a_{\nu} b^{\nu} = a_{\nu} g^{\nu\mu} b_{\mu}$$

Claim: $\tilde{A} \cdot \tilde{B} = A \cdot B$

Claim: $\boxed{\tilde{A}\cdot\tilde{B} = A\cdot B}$

proof: $\tilde{A} = L\cdot A$

$$\tilde{A}\bullet\tilde{B} = \tilde{A}^T G \tilde{B} = A^T \underbrace{L^T G L}_{G} B = A^T G B = A\bullet B \qquad \square$$

Corollary: $A^2 = A\bullet A = (a^0)^2 - \underline{a}^2$ is invariant!

Claim: $Q$ is a "four"-tensor

$\boxed{\operatorname{tr} Q \equiv Q^{\mu}{}_{\mu} = g_{\mu\nu} Q^{\mu\nu}}$ is invariant

proof: $L^T G L = G \quad\to\quad g_{\mu\nu} = L^{\mu'}{}_{\mu}\, g_{\mu'\nu'}\, L^{\nu'}{}_{\nu}$

$$\tilde{Q}^{\mu\nu} = L^{\mu}{}_{\mu'} L^{\nu}{}_{\nu'} Q^{\mu'\nu'}$$

$$\operatorname{tr}\tilde{Q} = g_{\mu\nu}\tilde{Q}^{\mu\nu} = \underbrace{g_{\mu\nu} L^{\mu}{}_{\mu'} L^{\nu}{}_{\nu'}}_{= g_{\mu'\nu'}} Q^{\mu'\nu'} = \operatorname{tr} Q \ \checkmark$$

L-invariant tensor!

other similar statements:

- $Q$ four-tensor, $A$ a four-vector

  $\Rightarrow \quad b^{\mu} \equiv Q^{\mu\nu} a_{\nu} = Q^{\mu\nu} g_{\nu\rho} a^{\rho} = Q^{\mu}{}_{\rho} a^{\rho} = \dots$ is a four-vector

- $Q$ has different representations:

  $Q^{\mu\nu}, \ Q_{\mu}{}^{\nu}, \ Q^{\mu}{}_{\nu}, \ Q_{\mu\nu}$

  - $Q^{\mu\nu} = g^{\mu\rho} Q_{\rho}{}^{\nu} = g^{\mu\rho} g^{\nu\eta} Q_{\rho\eta}$
  - $g^{\mu\nu} g_{\nu\rho} = g^{\mu}_{\rho} = \delta^{\mu}_{\rho} \quad\Rightarrow$

    $Q_{\rho}{}^{\nu} = g_{\rho\mu} Q^{\mu\nu}$

    $Q^{\mu}{}_{\rho} = Q^{\mu\nu} g_{\nu\rho}$

$g^{\mu\nu}$ and $g_{\mu\nu}$ move indices up and down!

Def: contractions

$T^{\mu\nu\varrho\cdots\eta} \Rightarrow T^{\mu\circ\cdots\eta}_{\ \circ}$ $(n-2)$ - tensor

$n$ - tensor

trivial example: $Q^{\mu\nu} = a^{\mu} \cdot b^{\nu}$

$\Rightarrow Q^{\circ}_{\circ} = \mathrm{tr}\, Q = a^{\mu} b_{\mu} = a \cdot b$ !

scalar !

Principle of relativity:

- no experiment can tell, which inertial frame of reference moves ...

$\Rightarrow$ Laws of Nature are L-invariant

~ can be formulated using "four-tensors"

Momentum and energy:

Newton: force $\Leftrightarrow$ momentum conservation!

(~ mass)

action - reaction

collision: $\exists\, m_1, m_2 \Rightarrow \boxed{m_1 \underline{v}_1 + m_2 \underline{v}_2 = m_1 \underline{v}_1' + m_2 \underline{v}_2'}$ (*)

$\underline{F}_{12} \equiv \frac{d}{dt}(m_2 \underline{v}_2) = - \underline{F}_{21} \equiv \frac{d}{dt}(m_1 \underline{v}_1)$

(*) is violated by velocity addition ...

$\exists$? four-vector replacing (*) ??

$\underline{v} = \frac{dx}{dt}$

$dx$ ← spatial part of $dx^{\mu}$ ~ four-vector

$dt$ ← not scalar

Proper time: $c^2 d\tau^2 = c^2 dt^2 - d\underline{x}^2$ ← L-invariant!

$$d\tau^2 = dt^2\left(1 - \frac{1}{c^2}\frac{d\underline{x}^2}{dt^2}\right) = dt^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\boxed{d\tau = dt\sqrt{1 - v^2/c^2}}$$

four-velocity: $\{u^\mu\} \equiv \left\{\frac{dx^\mu}{d\tau}\right\} = \underbrace{\frac{dt}{d\tau}}_{\frac{1}{(1-v^2/c^2)^{1/2}}}(c, \underline{v})$

$$\{u^\mu\} = \left(\frac{c}{\sqrt{1 - \underline{v}^2/c^2}}, \frac{\underline{v}}{\sqrt{1 - \underline{v}^2/c^2}}\right)$$ ← four-vector!

⇒ four-momentum:

$$p^\mu := m_0 u^\mu = \left(\frac{m_0 c}{\sqrt{1-\beta^2}}, \frac{m_0 \underline{v}}{\sqrt{1-\beta^2}}\right) =$$

rest mass

$$\boxed{\{p^\mu\} = m(c, \underline{v})}$$

relativistic mass $m = \frac{m_0}{\sqrt{1 - \underline{v}^2/c^2}}$

$p^\mu$ is conserved!

$$\boxed{p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu}$$

energy and momentum conservation

we want to extend Newton's 3d law...

Force $\boxed{\underline{F} = \frac{d}{dt}(m\underline{v})}$ action ~ reaction ... ✓

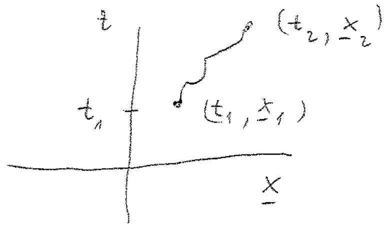
notice:

$$\underline{F} = \frac{d}{dt}\left(\frac{m_0}{(1 - v^2/c^2)^{1/2}}\underline{v}\right) = m \cdot \underline{\dot{v}} + \frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{\underline{v}\cdot\underline{\dot{v}}}{c^2}\cdot\underline{v}$$

$$= \underbrace{m\left(1 + \frac{\underline{v} \circ \underline{v}}{c^2 - v^2}\right)}_{\text{mass tensor!}} \cdot \underline{\dot{v}}$$

perpendicular $\quad \underline{F}_\perp = \frac{m_0}{\sqrt{1-v^2/c^2}} \cdot \dot{\underline{v}}_\perp \quad = m_\perp \dot{\underline{v}}_\perp$

paraller $\quad \underline{F}_\parallel = \frac{m_0}{(1-v^2/c^2)^{3/2}} \dot{\underline{v}}_\parallel \quad = m_\parallel \dot{\underline{v}}_\parallel$




Interpretation of $p^0$:

$$p^2 = m_0^2 c^2 = (p^0)^2 - \underline{p}^2$$

$$\Rightarrow \underbrace{p^0 \dot{p}^0}_{mc \cdot \dot{p}^0} = \underbrace{\underline{p}}_{m\underline{v}} \cdot \underbrace{\dot{\underline{p}}}_{\underline{F}} \quad \Rightarrow \quad \int_{t_1}^{t_2} \dot{p}^0 dt = \Delta p^0 = \frac{1}{c} \underbrace{\int dt\, \underline{F} \cdot \underline{v}}_{\int d\underline{x} \cdot \underline{F} = W}$$

$$\Delta p^0 = p^0(t_2) - p^0(t_1) = \frac{1}{c} W_{1\to 2}$$

$$\Rightarrow \quad \underbrace{p_0 = \frac{1}{c} E}\ + \text{cst}$$

$$\boxed{\{p^\mu\} = \left(\frac{E}{c}, \underline{p}\right)} \quad \Rightarrow \quad \boxed{E = c p^0 = m \cdot c^2 = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}}}$$

Einstein

expand $\quad E = \underbrace{m_0 c^2}_{\text{rest energy}} + \underbrace{\frac{1}{2} m_0 v^2 + \ldots}_{\text{kinetic energy}}$

conservation of four-mom. $\Longleftrightarrow$ energy + momentum conservation!

energy - momentum relation:

$$p^2 = m_0^2 c^2 = \frac{E^2}{c^2} - \underline{p}^2 \quad \Rightarrow \quad \boxed{E = \pm \sqrt{m_0^2 c^4 + \underline{p}^2 c^2}}$$

$|\underline{p}| \ll m_0 c$

$$E = m_0 c^2 + \frac{\underline{p}^2}{2m_0} + \ldots$$

$|\underline{p}| \gg m_0 c$

$$E = |\underline{p}| \cdot c$$

~ photon

Mass deficiency:

consider binding of n and p :

$$\boxed{n + p \rightarrow D + \gamma}$$


momentum balance $\quad M_D \cdot \underline{V} = \hbar \underline{k}$

energy balance $\quad c^2 m_n^0 + c^2 m_p^0 = \underbrace{c^2 M_D}_{c^2 M_D^0 + \cancel{\frac{1}{2} M_D^0 V^2 + \dots}} + \hbar\omega$

small

$\Rightarrow \quad c^2 m_n^0 + c^2 m_p^0 \simeq c^2 M_D^0 + \hbar\omega$

takes away binding energy!

$$\boxed{M_D^0 = m_n^0 + m_p^0 + \frac{1}{c^2}\Delta E}$$

binding energy $\hat{=} - \hbar\omega$

$\Rightarrow \quad \underline{M_D^0 < m_n^0 + m_p^0}$ ! ~ mass deficiency

Dalton

Check : $\quad m_n^0 c^2 = 1{,}0078 \cdot u$ $\quad$ ($10^9$ x chemical reaction!)

$= 0{,}9395$ GeV

$m_p^0 c^2 = 1{,}0086 \cdot u = 938{,}27$ MeV

$= 0{,}93827$ GeV

$M_D^0 c^2 = 2{,}0141$ u

$$\boxed{M_D^0 = m_p^0 + m_n^0 - 2{,}16 \text{ MeV}}$$

Other example:

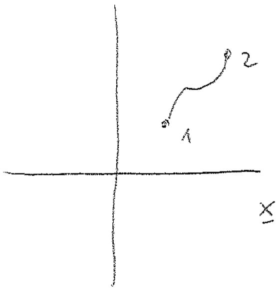
$\beta^-$ decay: $\quad \boxed{n \xrightarrow{\sim 15 \text{ min}} p + e + \bar{\nu}_e}$

511 keV

240 keV !

Lagrangian formulation:

free particle



classical Lagrangian

$$L = \frac{m_0}{2} \dot{\underline{x}}^2$$

$$S_{cl} = \int_{t_1}^{t_2} dt \; \frac{m_0}{2} \dot{\underline{x}}^2 \quad \Rightarrow \text{ straight line } \ddot{\underline{x}} = \phi$$

Lorentz - invariant action?

$$\overset{?}{\Rightarrow} \quad S \sim \int_1^2 ds \sim \int_1^2 d\tau \sim \int_{t_1}^{t_2} dt \sqrt{1 - v^2/c^2}$$

$$S = -\alpha \int dt \left(1 - \dot{\underline{x}}^2/c^2\right)^{1/2} \quad \xrightarrow[c \to \infty]{} \quad \int_{t_1}^{t_2} \frac{\alpha}{2} \frac{\dot{\underline{x}}^2}{c^2} - \alpha(t_2 - t_1)$$

$$\Rightarrow \alpha = m_0 c^2$$

$$\boxed{S = -m_0 c^2 \int dt \sqrt{1 - \dot{\underline{x}}^2/c^2}}$$

momentum $\quad p = \frac{\partial L}{\partial \dot{\underline{x}}} = \frac{m_0}{\sqrt{1 - \dot{\underline{x}}^2/c^2}} \dot{\underline{x}} = m \cdot \underline{v} \quad \checkmark \quad (a)$

energy:

$$\mathcal{H} = p \dot{\underline{x}} - L = \frac{m_0 \dot{\underline{x}}^2}{\sqrt{1 - v^2/c^2}} + m_0 c^2 \sqrt{1 - \dot{\underline{x}}^2/c^2} =$$

$$= \frac{m_0 \cdot c^2}{\sqrt{1 - \dot{\underline{x}}^2/c^2}} \quad (b)$$

$\Rightarrow$ invert (a) $\Rightarrow$ (b)

$$\boxed{\mathcal{H}(p) = \sqrt{c^2 p^2 + m_0^2 c^4}}$$

external force?

Search for four-vectors:

$$A \nearrow a^\mu \quad \text{contravariant coordinates}$$
$$A \searrow a_\mu \quad \text{covariant}$$

$\uparrow$ vector

L-transformation: $\boxed{\tilde{a}^\nu = \Lambda^\nu{}_\mu a^\mu}$

$$\Rightarrow \quad \tilde{a}_\nu = g_{\nu\varrho} \tilde{a}^\varrho = g_{\nu\varrho} \Lambda^\varrho{}_\eta a^\eta = \underbrace{g_{\nu\varrho} \Lambda^\varrho{}_\eta g^{\eta\mu}}_{\Lambda_\nu{}^\mu a_\mu} a_\mu$$

multiply by $\Lambda^T$ !

$$\Lambda^\nu{}_\kappa \tilde{a}_\nu = \underbrace{\Lambda^\nu{}_\kappa g_{\nu\varrho} \Lambda^\varrho{}_\eta}_{\Lambda^T G \Lambda = G \Rightarrow g_{\kappa\eta}} g^{\eta\mu} a_\mu = \underbrace{g_{\kappa\eta} g^{\eta\mu}}_{\delta^\mu} a_\mu = a_\kappa$$

$$\Rightarrow \quad \boxed{\tilde{a}_\nu = \Lambda_\nu{}^\mu a_\mu} \quad \text{or} \quad \boxed{\Lambda^\nu{}_\mu \tilde{a}_\nu = a_\mu}$$

Statement: $A$ is four-vector and $\exists\ (b_0, b_1, b_2, b_3)$ such that $A \cdot B = a^\mu b_\mu = \tilde{a}^\mu \tilde{b}_\mu$ always

$\Rightarrow$ $\{b_\mu\}$ are covariant components of a four-vector!

Proof: $\tilde{a}^\mu \tilde{b}_\mu = \Lambda^\mu{}_\nu a \tilde{b}_\mu = a^\nu b_\nu$

for <u>any</u> setup $\Rightarrow \Lambda^\mu{}_\nu \tilde{b}_\mu = b_\nu$ ✓

- four-wave-vector:

consider propagation of electromagnetic wave

$\sim \sin(\omega t - \underline{k} \cdot \underline{r}) \sim$ nodes should be same for all observers!

$$\Rightarrow \quad (ct, \underline{r}) \cdot \begin{pmatrix} \omega/c \\ \\ -\underline{k} \end{pmatrix} \quad \text{is invariant (L-invariant)}$$

$$\Rightarrow \quad \{k_\mu\} \equiv \left(\frac{\omega}{c}, -\underline{k}\right) \quad \text{is a four-vector}$$

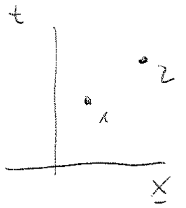
$$\boxed{\{k^\mu\} = \left(\frac{\omega}{c}, \underline{k}\right)}$$

$$\boxed{\{\hbar k^\mu\} = \left(\underbrace{\frac{\hbar\omega}{c}}_{\text{energy}/c}, \underbrace{\hbar \underline{k}}_{\text{momentum of photon}}\right)} \quad ✓$$

- <u>differentiation</u> ? → <u>gradient</u> ??

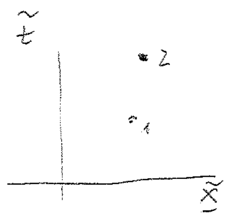
scalar function $f(x) \longrightarrow f(\tilde{x}) = f(x)$

other observer — me



$$f(x_2) - f(x_1) = f(\tilde{x}_2) - f(\tilde{x}_1) \quad !$$

$$\Downarrow$$



$$dx^\mu \frac{\partial f}{\partial x^\mu} = d\tilde{x}^\mu \frac{\partial f}{\partial \tilde{x}^\mu}$$

covariant components of a four-vector !

$$\{\partial_\mu f\} = \left\{\frac{\partial f}{\partial x^\mu}\right\} = \left(\frac{1}{c}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial \underline{x}}\right)$$

<u>gradient</u>: $\boxed{\{\partial^\mu\} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right)}$

$$\Rightarrow \text{Scalar}: \quad \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta = \Box$$

d'Alembert

• Force as four-vector ??

$$\underline{F} = \frac{d\underline{p}}{dt} \quad \leftarrow \text{ not piece of four-vector...}$$

$$\boxed{f^\mu \equiv \frac{dp^\mu}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}\left(\frac{1}{c}\dot{E},\ \underline{F}\right)}$$

four-force $\qquad\qquad\qquad\qquad \dot{\underline{p}}$

"conservative" force : $f^\mu = \partial^\mu U(x)$ ?

charged particle ?

• current density :

continuity equ: $\frac{\partial \rho}{\partial t} + \operatorname{div} \underline{j} = 0$

$$\Rightarrow \boxed{j^\mu \equiv (c\rho, \underline{j})} \Rightarrow \boxed{\partial_\mu j^\mu = 0}$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

## Maxwell equations, Lorentz-force:

Maxwell:

$$\operatorname{div} \underline{E} = \frac{1}{\epsilon_0}\rho$$

$$\operatorname{div} \underline{B} = 0$$

$$\operatorname{rot} \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\operatorname{rot} \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

vector potential :

$$\operatorname{div} \underline{B} = 0 \Rightarrow \exists\, \underline{A} \quad \boxed{\underline{B} = \operatorname{rot} \underline{A}}$$

$$\Rightarrow \operatorname{rot}(\underline{E} + \dot{\underline{A}}) = 0 \Rightarrow \exists\, \phi : \boxed{\underline{E} = -\operatorname{grad}\phi - \dot{\underline{A}}}$$

measurable

3 + 3 fields $\Leftrightarrow$ 1 + 3 $\leftarrow$ more fundamental

+ redundancy !

Gauge freedom:

$$\underline{A} \to \underline{A} + \operatorname{grad} f \quad \Rightarrow \quad \underline{B} \text{ same}$$

$$\phi \to \phi - \dot{f} \quad \Rightarrow \quad \underline{E} \text{ same!}$$

$f(t, \underline{x})$ arbitrary

$$\partial^\mu f = \left(\frac{1}{c}\frac{\partial f}{\partial t}, -\operatorname{grad} f\right) \quad \sim \text{four-vector}$$

$$\Rightarrow \quad \boxed{A^\mu \equiv \left(\frac{1}{c}\phi, \underline{A}\right)}$$

gauge transformation:

$$A^\mu \to A^\mu - \partial^\mu f$$

$A^\mu$ generates $\underline{E}$ and $\underline{B}$!

$$\boxed{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu} \quad \Rightarrow \quad \text{tensor! contains } \underline{E}, \underline{B}$$

$$F^{0i} = \frac{1}{c}\frac{\partial A^i}{\partial t} + \frac{\partial}{\partial x^i}\frac{\phi}{c} = -\frac{1}{c}E^i$$

$$F^{ij} = -\frac{\partial A^j}{\partial x^i} + \frac{\partial A^i}{\partial x^j} \qquad F^{12} = -(\operatorname{rot} A)^3 = -B^3$$

$$\{F^{\mu\nu}\} = \left(\begin{array}{c|ccc} & & -\frac{1}{c}\underline{E} & \\ \hline & & -B^3 & B^2 \\ \frac{1}{c}\underline{E} & B^3 & & B^1 \\ & -B^2 & B^1 & \end{array}\right) \longleftarrow \quad \underline{B}\times$$

$$\{F^\mu{}_\nu\} = \left(\begin{array}{c|c} & E/c \\ \hline E/c & -\underline{B}\times \end{array}\right)$$

$$\partial_\mu F^{\mu\nu} = \left(\frac{1}{c}\operatorname{div}\underline{E}, -\frac{1}{c^2}\dot{\underline{E}} + \operatorname{rot}\underline{B}\right) = \left(\frac{1}{c}\frac{1}{\epsilon_0}\rho, \mu_0 \underline{j}\right) = \mu_0(c\rho, \underline{j})$$

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 j^\nu}$$

L-invariant action ?

$$\sim \int d\tau \, U^\mu \cdot A_\mu = \int dx_\mu \, A^\mu =$$

$$= \int c \cdot dt \, \frac{1}{c} \phi - \int d\underline{x} \cdot \underline{A} = \int dt \left( \phi - \dot{\underline{x}} \cdot \underline{A} \right)$$

potential !

$\Rightarrow$

$$\boxed{L(\underline{x}, \dot{\underline{x}}, t) = L \equiv -m_0 c^2 \sqrt{1 - \dot{x}^2/c^2} - e\,\phi(\underline{x},t) + e\, \dot{\underline{x}} \cdot \underline{A}(\underline{x}, t)}$$

equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\underline{x}}} = \frac{\partial L}{\partial \underline{x}}$$

$$\frac{d}{dt} \left( \frac{m_0 \dot{\underline{x}}}{\sqrt{1 - \dot{x}^2/c^2}} + e \underline{A} \right) = -e \, \mathrm{grad}\, \phi + e \frac{\partial}{\partial \underline{x}} \left( \dot{\underline{x}} \cdot \underline{A} \right)$$

$$\downarrow$$

$$e \dot{A}^i + e \frac{\partial A^i}{\partial x^j} \dot{x}^j$$

$$\frac{d}{dt}(p^i) = -e \left( \frac{\partial \phi}{\partial x^i} + \dot{A}^i \right) + e \left( \frac{\partial A^k}{\partial x^i} v^k - \frac{\partial A^i}{\partial x^k} v^k \right)$$

$$e \underline{E} + e \underline{v} \times \underline{B} \quad !$$

claim:

$$\boxed{\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu U^\nu}$$

homework: prove it!

$$\mu = 0 \Rightarrow \frac{1}{c} \dot{E} = q \frac{1}{c} \underline{E} \cdot \underline{v} \quad \checkmark$$

$$\mu = 1, .., 3 \Rightarrow \frac{dp}{dt} = q \underline{E} - q \underline{B} \times \underline{v} \quad \checkmark$$