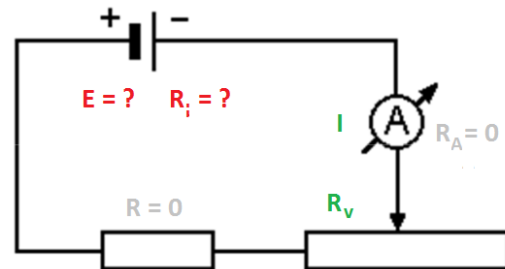
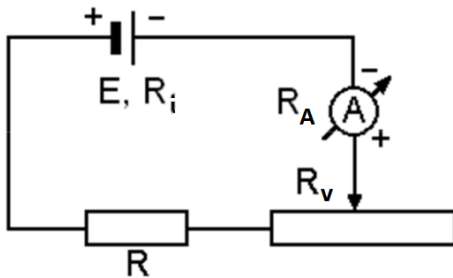


CURVE FITTING

In a series circuit we adjust the resistance and measure the current.

Real measurement: source (E , R_i), variable resistor (R_v), constant resistor (R), ammeter (R_A). Here we neglect R and R_A .



Data

$R_v (\Omega)$	$I (A)$
100	0.082
200	0.044
250	0.036
300	0.030
400	0.022
600	0.016

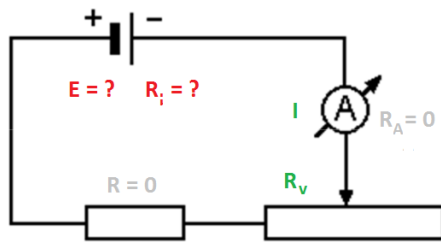
We want to know the value of E and R_i .

Excel trendline:

$$y = 5.8536 x^{-0.925}$$

$$I = 5.8536 R_v^{-0.925} \rightarrow E = ?!?, R_i = ?!?$$

Only useful for iteration in this interval.

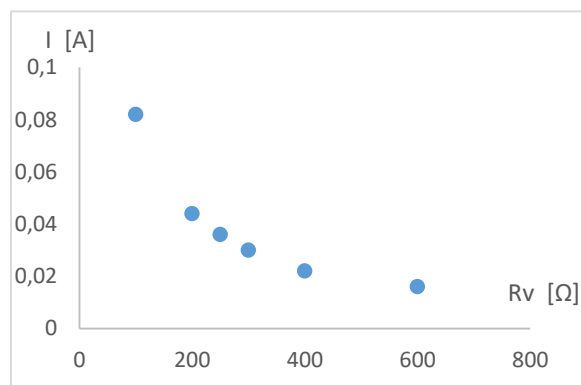


$$E = I (R_i + R_v) \rightarrow$$

$$I = \frac{E}{R_i + R_v}$$

x	R_v
y	I
A	E
B	R_i

$$y = \frac{A}{B + x}$$



From theory we know that we have a hyperbola.

$$I = \frac{E}{R_i + R_v}$$

$R_v (\Omega)$	$I (A)$
100	0.082
200	0.044
250	0.036
300	0.030
400	0.022
600	0.016

$$0.082 = \frac{E}{R_i + 100} \quad (1)$$

$$0.044 = \frac{E}{R_i + 200} \quad (2)$$

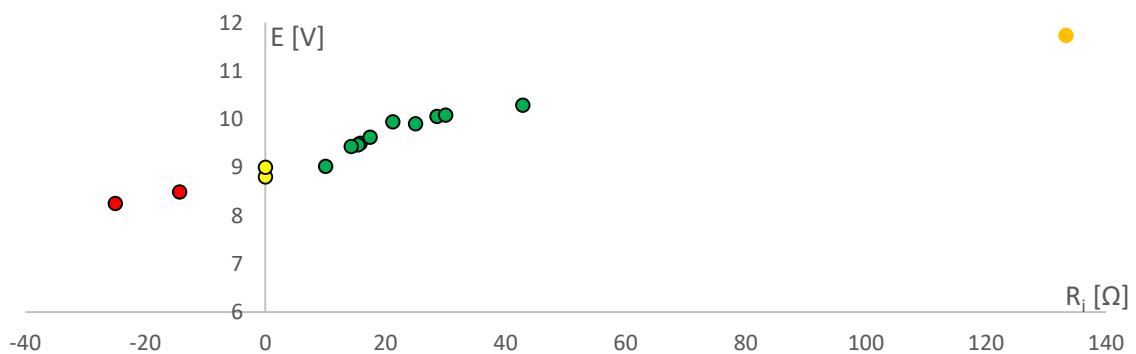
$$0.036 = \frac{E}{R_i + 250} \quad (3)$$

$$0.030 = \frac{E}{R_i + 300} \quad (4)$$

$$0.022 = \frac{E}{R_i + 400} \quad (5)$$

$$0.016 = \frac{E}{R_i + 600} \quad (6)$$

6 equations, 2 unknowns: 15 equation systems!



arithmetic mean of the 15 solutions:

$$E = 9.571 \text{ V}$$

$$R_i = 20.97 \Omega$$

is this good ?!?

$$E = 9.57082 \approx 9.57 \text{ V}; \quad R_i = 20.9693 \approx 21 \, \Omega$$

$$I = \frac{E}{R_i + R_v} : \quad y = \frac{9.57}{21 + x}$$

MAKHBER

mark data, Plot → Scatter

Graph → Add Function Ctrl + Alt + F

$$f(x) = 9.57 / (x + 21)$$

From x = 0 → 90

To x = 1 → 600

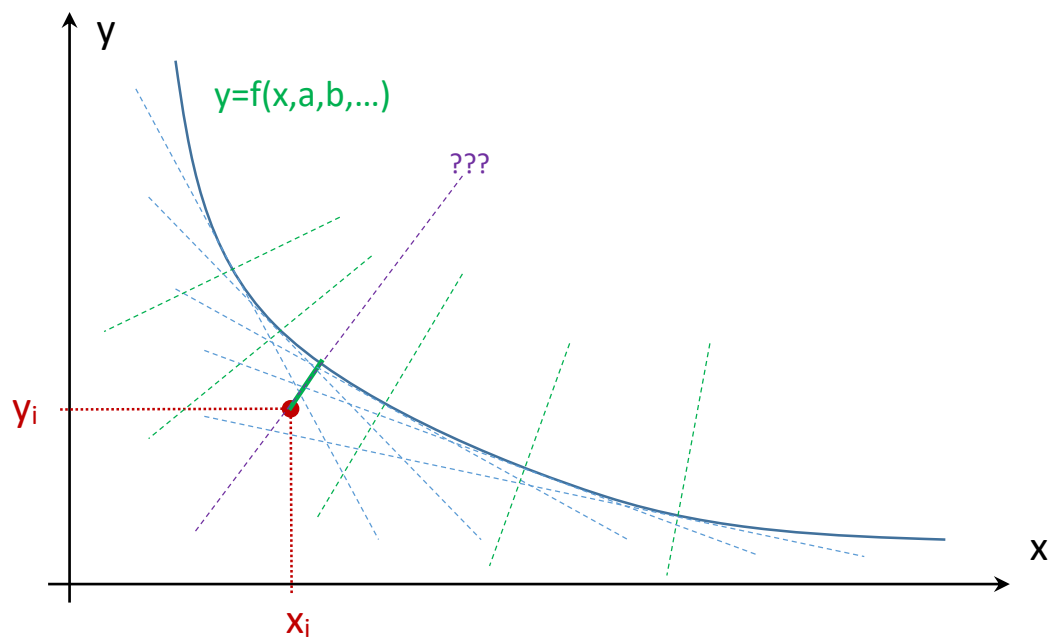
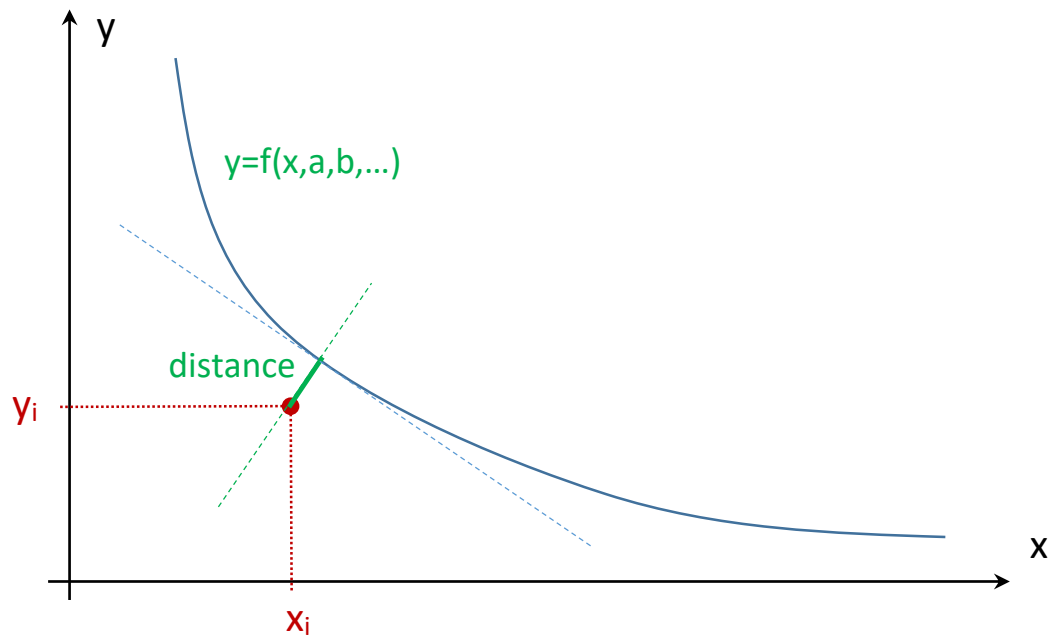
Points 100

Ok

better fit?

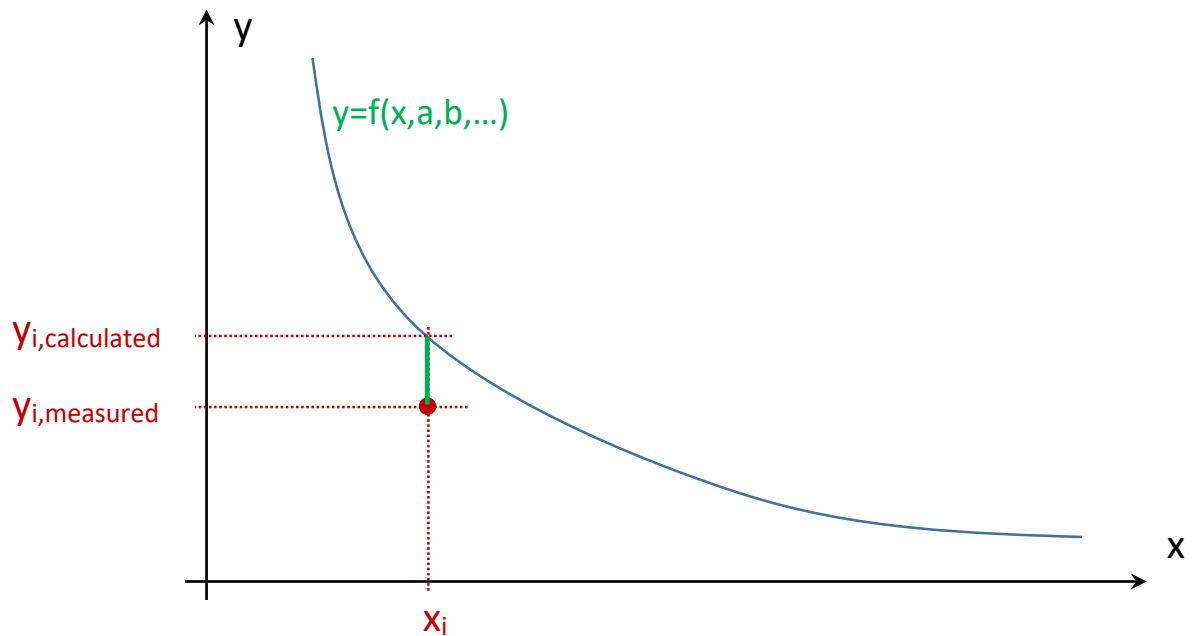
E = ??? R_i = ???

We want minimum distance:



It is not possible to express the distance as a function of the unknown parameters.

The distance will be redefined this way:



$$S = \sum (y_{i,calculated} - y_{i,measured})^2 \rightarrow \min.$$

$$S = \sum (f(x_i,a,b,...) - y_{i,measured})^2 \rightarrow \min.$$

“least squares method”

$$y = \frac{A}{x+B} :$$

$$S = \sum \left(\frac{A}{x_i + B} - y_i \right)^2 \rightarrow \min.$$

$$\frac{\partial S}{\partial A} = 2 \sum \left(\frac{A}{x_i + B} - y_i \right) \cdot \left(\frac{1}{x_i + B} \right) = 0$$

$$\frac{\partial S}{\partial B} = 2 \sum \left(\frac{A}{x_i + B} - y_i \right) \cdot \left(-\frac{A}{(x_i + B)^2} \right) = 0$$

i	x_i	y_i
1	100	0.082
2	200	0.044
3	250	0.036
4	300	0.030
5	400	0.022
6	600	0.016

→ numerical solution gives

$$E = 9.480524812 \text{ V} \quad R_i = 15.59110420 \text{ } \Omega$$

$$E \approx 9.48 \text{ V} \quad R_i \approx 15.6 \text{ } \Omega$$

$$I = \frac{E}{R_i + R_v} : \quad y = \frac{9.48}{15.6 + x} \quad \text{Makhber}$$

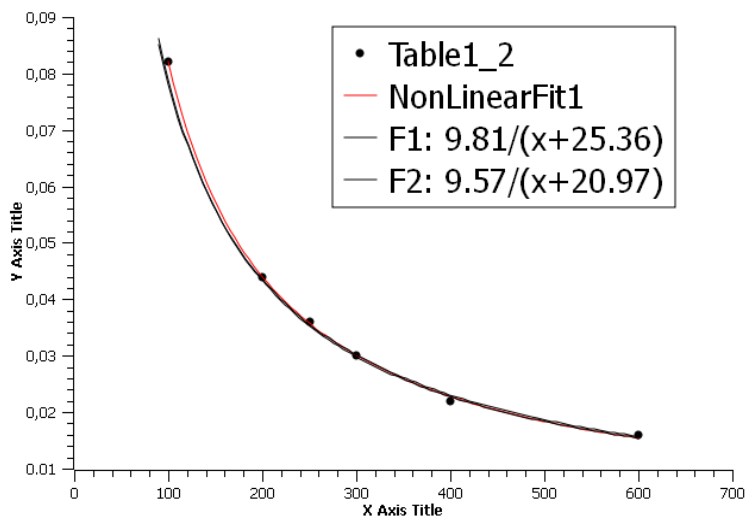
Makhber:

Analysis → Fit wizard Ctrl + Y

$a / (x + b)$

Fit >>

Fit



a: $E = (9.4805247987374 \pm 0.131725431900025) \text{ V}$

b: $R_i = (15.5911037006196 \pm 1.97162054657795) \Omega$

$\chi^2/\text{doF} = 2.78453858417807\text{e-}07$

$R^2 = 0.999599250863395$

$E = (9.48 \pm 0.132) \text{ V}$

$R_i = (15.6 \pm 1.97) \Omega$

relative error:

$$\frac{\Delta E}{E} = \frac{0.132}{9.48} = 0.01389 = 1.389\%$$

$$\frac{\Delta R_i}{R_i} = \frac{1.971}{15.59} = 0.1265 = 12.65\%$$

Excel solver!!!

LINEAR: $y = ax + b$

$$S = \sum(ax_i + b - y_i)^2 \rightarrow \min.$$

$$\frac{\partial S}{\partial a} = 2 \sum(ax_i + b - y_i) \cdot x_i = 0 \quad (1)$$

$$\frac{\partial S}{\partial b} = 2 \sum(ax_i + b - y_i) \cdot 1 = 0 \quad (2)$$

$$\begin{aligned} (1): \quad \sum(ax_i + b - y_i) \cdot x_i &= \sum(ax_i^2 + bx_i - x_i y_i) = \\ &= \sum(ax_i^2) + \sum(bx_i) + \sum(-x_i y_i) = \\ &= a \sum(x_i^2) + b \sum(x_i) - \sum(x_i y_i) = 0 \end{aligned}$$

$$\begin{aligned} (2): \quad \sum(ax_i + b - y_i) &= \sum(ax_i) + \sum(b) + \sum(-y_i) = \\ &= a \sum(x_i) + n \cdot b - \sum(y_i) = 0 \\ \rightarrow \quad b &= \frac{\sum(y_i) - a \sum(x_i)}{n} \end{aligned}$$

$$\begin{aligned} (1): \quad a \sum(x_i^2) + \frac{\sum(y_i) - a \sum(x_i)}{n} \cdot \sum(x_i) - \sum(x_i y_i) &= 0 \\ a \sum(x_i^2) + \frac{\sum(y_i) \cdot \sum(x_i)}{n} - \frac{a \sum(x_i) \cdot \sum(x_i)}{n} - \sum(x_i y_i) &= 0 \\ a \left(\sum(x_i^2) - \frac{\sum(x_i) \cdot \sum(x_i)}{n} \right) &= \sum(x_i y_i) - \frac{\sum(y_i) \cdot \sum(x_i)}{n} \\ a = \frac{\sum(x_i y_i) - \frac{\sum(y_i) \cdot \sum(x_i)}{n}}{\sum(x_i^2) - \frac{\sum(x_i) \cdot \sum(x_i)}{n}} & \quad a = \frac{\frac{\sum(x_i y_i)}{n} - \frac{\sum(x_i)}{n} \cdot \frac{\sum(y_i)}{n}}{\frac{\sum(x_i^2)}{n} - \frac{\sum(x_i)}{n} \cdot \frac{\sum(x_i)}{n}} \end{aligned}$$

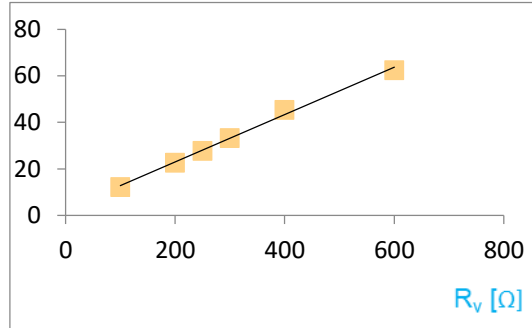
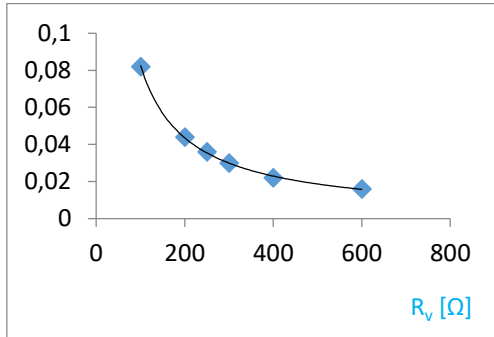
$$a = \frac{\overline{x \cdot y} - \bar{x} \cdot \bar{y}}{\overline{x^2} - \bar{x}^2} \quad \text{and} \quad b = \bar{y} - a \cdot \bar{x}$$

LINEARIZATION

$$I = \frac{E}{R_v + R_i}$$



$$f(I) = a \cdot R_v + b \quad ????$$



$$\frac{1}{I} = \frac{R_v + R_i}{E}$$

$$\frac{1}{I} = \frac{1}{E} \cdot R_v + \frac{R_i}{E}$$

$$y' = a \cdot x' + b$$

$$x' = x = R_v$$

$$y' = \frac{1}{I}$$

$$a = \frac{1}{E}$$

$$b = \frac{R_i}{E}$$

Linearized formula:

$$\frac{1}{I} = \frac{1}{E} \cdot R_V + \frac{R_i}{E}$$

$$y = a \cdot x + b$$

slope:

$$a = \frac{\overline{x \cdot y} - \bar{x} \cdot \bar{y}}{\overline{x^2} - \bar{x}^2} : \quad x = R_V; \quad y = \frac{1}{I} \quad \rightarrow \quad a = \frac{\left(\overline{R_V \cdot \frac{1}{I}} \right) - \bar{R}_V \cdot \left(\overline{\frac{1}{I}} \right)}{\left(\overline{R_V^2} \right) - \bar{R}_V^2} \quad a = 0.101884$$

Units!

$$\bar{x} = \bar{R}_V [\Omega]; \quad \bar{y} = \left(\overline{\frac{1}{I}} \right) [1/A]; \quad \overline{x^2} = \left(\overline{R_V^2} \right) [\Omega^2]; \quad \overline{x \cdot y} = \left(\overline{R_V \cdot \frac{1}{I}} \right) [\Omega/A] \quad a = 0.101884 \text{ 1/V}$$

$$a = \frac{1}{E} \quad \rightarrow \quad E = 1/a = 9.815049 \text{ V} = 9.815 \text{ V}$$

intercept:

$$b = \bar{y} - a \cdot \bar{x} : \quad x = R_V; \quad y = \frac{1}{I} \quad \rightarrow \quad b = \left(\overline{\frac{1}{I}} \right) - a \cdot \bar{R}_V \quad b = 2.583663 \text{ 1/A}$$

$$b = \frac{R_i}{E} \quad \rightarrow \quad R_i = b/a = 25.35878 \Omega = 25.36 \Omega$$

Excel functions + trendline

=slope(y-values;x-values)

=intercept(y-values;x-values)

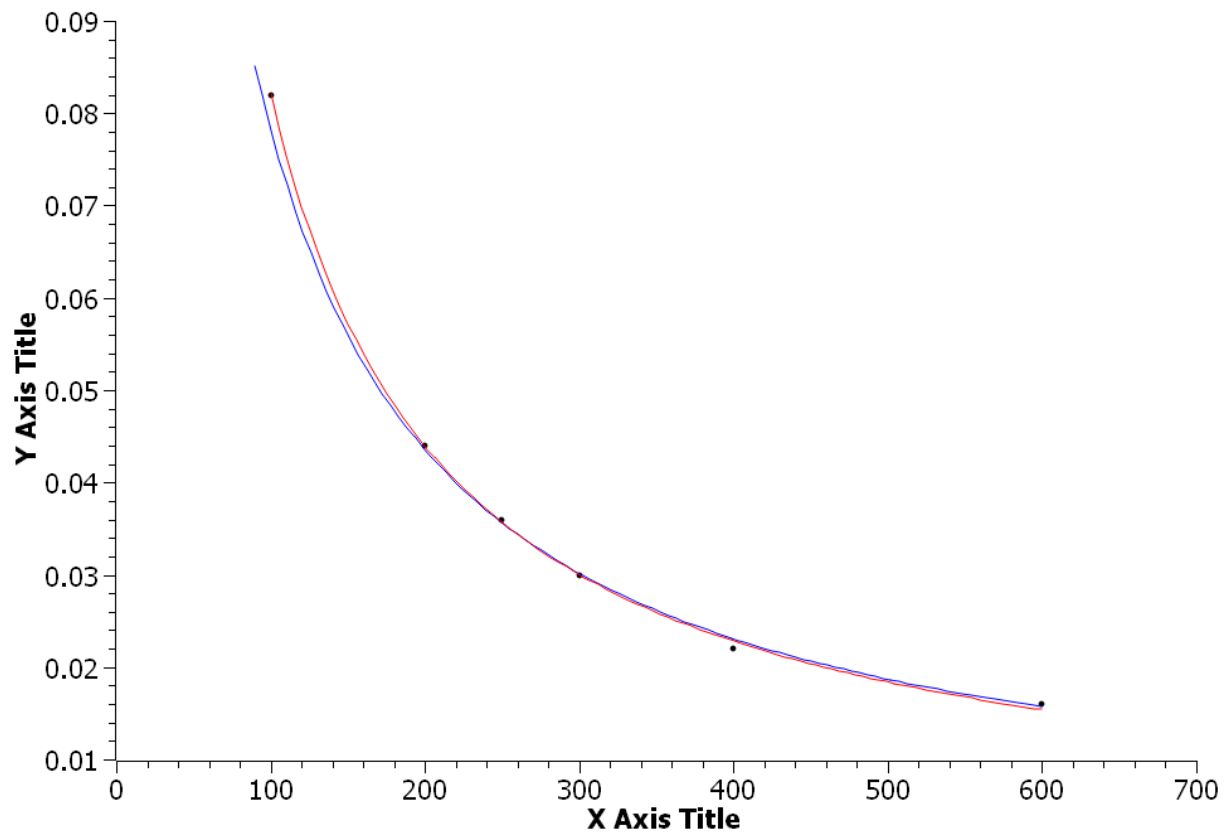
linearization: $E = 9.815 \text{ V}$

$R_i = 25.36 \Omega$

hiperbola: $E = (9.48 \pm 0.132) \text{ V}$

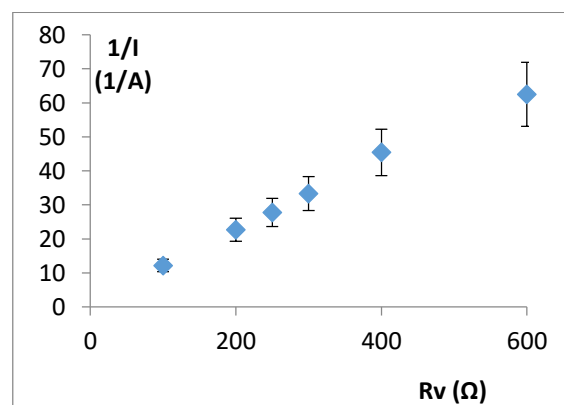
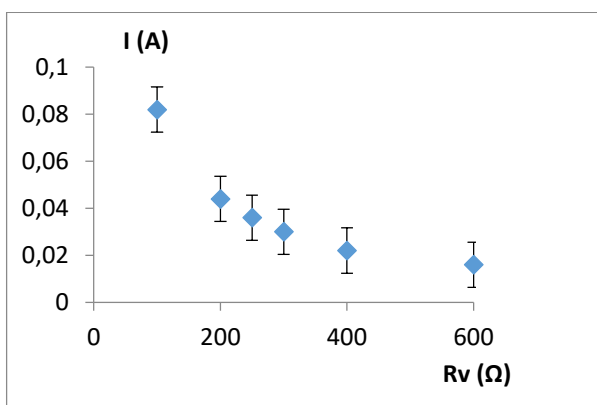
$R_i = (15.6 \pm 1.97) \Omega$

why do we get different results?!?



hiperbola
 (Makhber)
 $E = (9.48 \pm 0.132) \text{ V}$
 $R_i = (15.6 \pm 1.97) \Omega$
 $R^2 = 0.9996$

linearization
 (Excel)
 $E = (9.82 \pm ???) \text{ V}$
 $R_i = (25 \pm ???) \Omega$
 $R^2 = 0.9959$



because of the linearization!
 it is better to use nonlinear fitting if possible

Standard deviation of the slope and the intercept

s_r residual standard deviation

$$s_r^2 = \frac{\sum (a \cdot x_i + b - y_i)^2}{N-2}$$

$$\text{Var}[a] = \frac{s_r^2}{N \cdot (\overline{x^2} - \bar{x}^2)} \rightarrow s_a = \sqrt{\text{Var}[a]}$$

$$\text{Var}[b] = \bar{x}^2 \cdot \text{Var}[a] \rightarrow s_b = \sqrt{\text{Var}[b]}$$

Excel:

$$(a \cdot x_i + b - y_i)^2 : \left(a \cdot R_v + b - \frac{1}{I} \right)^2 \rightarrow \text{sum} \rightarrow s_r^2 = \dots$$

$$s_r^2 = 1.6137 \text{ 1/A}^2$$

$$\text{Var}[a] = 1.061\text{E-}05 \text{ 1/V}^2 \quad s_a = 0.00326 \text{ 1/V}$$

$$\text{Var}[b] = 1.2776902 \text{ 1/A}^2 \quad s_b = 1.13035 \text{ 1/A}$$

$$a = (0.101884 \pm 0.00326) \text{ 1/V}$$

$$b = (2.583663 \pm 1.13035) \text{ 1/A}$$

Error interval

E and R_i are calculated from a and b.

$$a = (0.101884 \pm 0.00326) \text{ 1/V}$$

$$b = (2.583663 \pm 1.13035) \text{ 1/A}$$

$$a = \frac{1}{E} \rightarrow E = \frac{1}{a}$$

$$\Delta E = \sqrt{\left(\frac{\partial E}{\partial a} \cdot \Delta a\right)^2 + \left(\frac{\partial E}{\partial b} \cdot \Delta b\right)^2} = \sqrt{\left(-\frac{1}{a^2} \cdot \Delta a\right)^2 + 0} = \left|\frac{1}{a^2} \cdot \Delta a\right| = 0.3138 \text{ V}$$

$$b = \frac{R_i}{E} \rightarrow R_i = \frac{b}{a}$$

$$\Delta R_i = \sqrt{\left(\frac{\partial R_i}{\partial a} \cdot \Delta a\right)^2 + \left(\frac{\partial R_i}{\partial b} \cdot \Delta b\right)^2} = \sqrt{\left(-\frac{b}{a^2} \cdot \Delta a\right)^2 + \left(\frac{1}{a} \cdot \Delta b\right)^2} = 11.124 \text{ } \Omega$$

$$E = (9.82 \pm 0.31) \text{ V}$$

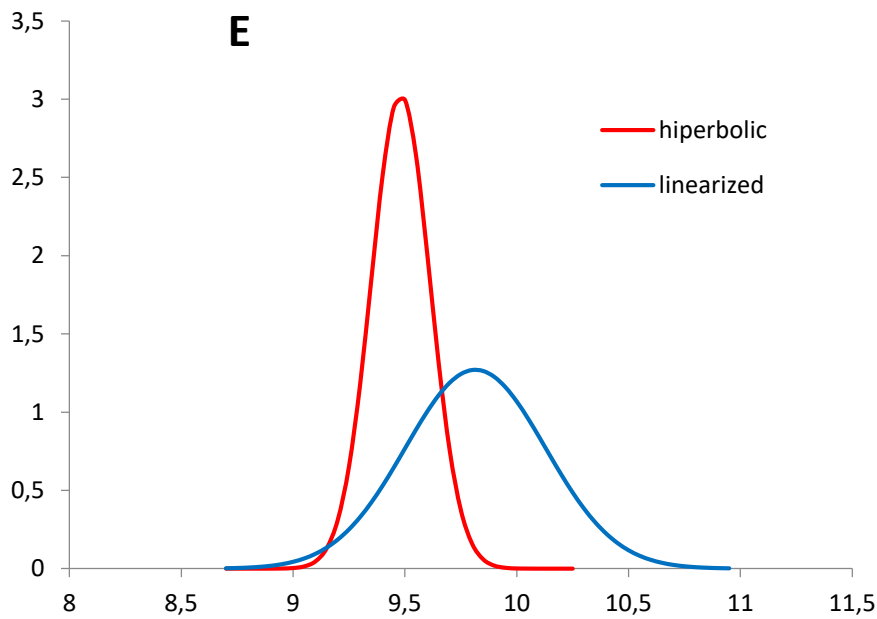
$$R_b = (25.4 \pm 11.1) \text{ } \Omega$$

SUMMARY

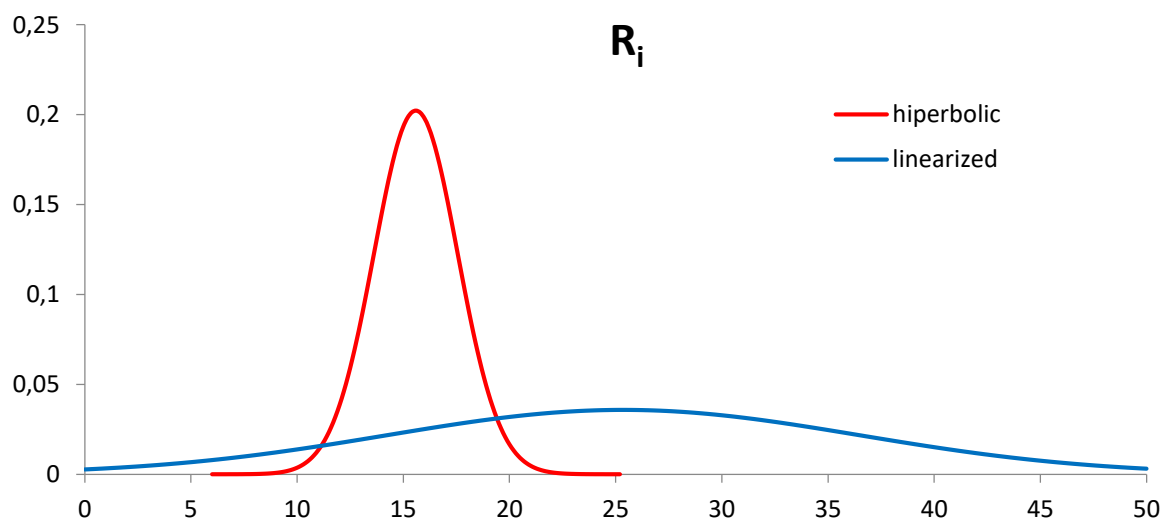
Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

E electromotive force: $E = (9.48 \pm 0.132) \text{ V}$ $E = (9.82 \pm 0.31) \text{ V}$



R_i internal resistance: $R_i = (15.6 \pm 1.97) \Omega$ $R_i = (25 \pm 11.1) \Omega$



In case of a **linear function** $y = a x$ (the intercept of the line is zero)

$$S(a) = \sum_{i=1}^N (ax_i - y_i)^2 \rightarrow \min. \quad \text{leads to} \quad a = \frac{\overline{x \cdot y}}{\overline{x^2}}.$$

and the **standard deviation** of the slope can be calculated applying the following formula:

$$s_a = \sqrt{\frac{\sum_{i=1}^N (a \cdot x_i - y_i)^2}{N \cdot (N-1) \cdot \overline{x^2}}}$$