The finite element method (FEM)

A végeselem módszer alapjai

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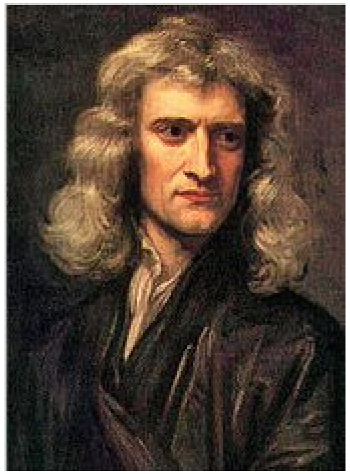
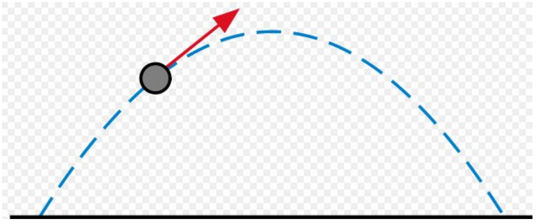
Solving Physics problems



Problems described in the „language” of Mathematics as differential equations

1. Ordinary differential equation (ODE)

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = F(x(t)),$$

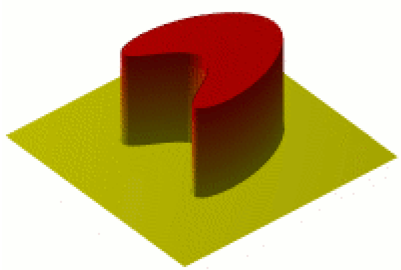


2. Partial differential equation (PDE)

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0$$

$$\mathbf{q} = -k\,\nabla u$$

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

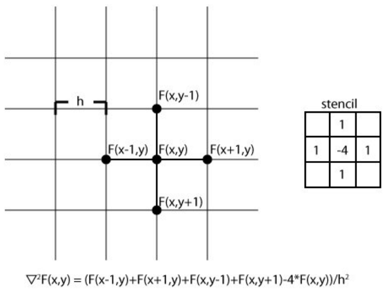


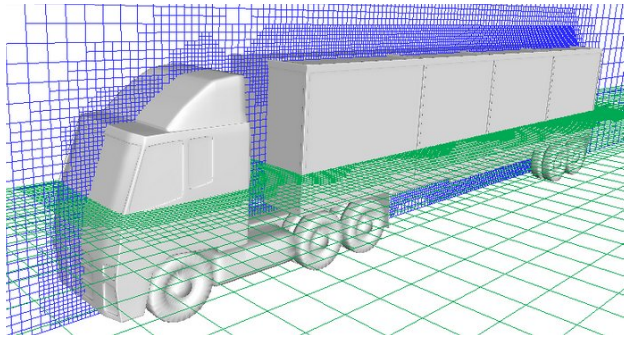
Heat equation

Differential equations

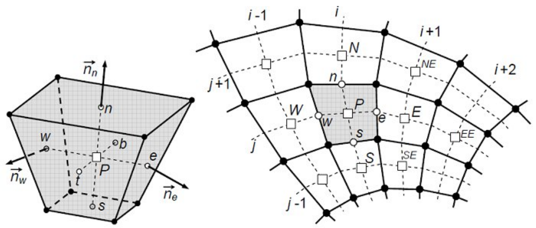
Solving PDE equations?

1. Finite difference method

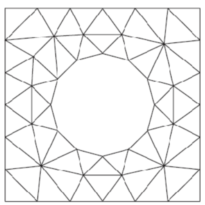




2. Finite volume method



3. Finite element method



The Finite Element Method (FEM) – A végeselem módszer alapjai és alkalmazásai 2015

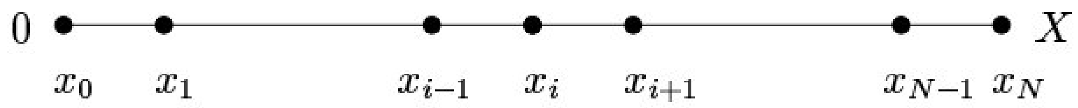
Finite difference method

Finite difference method

Principle: derivatives in the partial differential equation are approximated by linear combinations of function values at the grid points

1D: $\Omega = (0, X)$, $\quad u_i \approx u(x_i)$, $\quad i = 0, 1, \ldots, N$

grid points $\quad x_i = i\Delta x \qquad$ mesh size $\quad \Delta x = \frac{X}{N}$



First-order derivatives

$$\frac{\partial u}{\partial x}(\bar{x}) = \lim_{\Delta x \to 0} \frac{u(\bar{x} + \Delta x) - u(\bar{x})}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(\bar{x}) - u(\bar{x} - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(\bar{x} + \Delta x) - u(\bar{x} - \Delta x)}{2\Delta x} \qquad \text{(by definition)}$$

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Finite difference method

Approximation of second-order derivatives

Central difference scheme

$$T_1 + T_2 \quad \Rightarrow \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x)^2$$

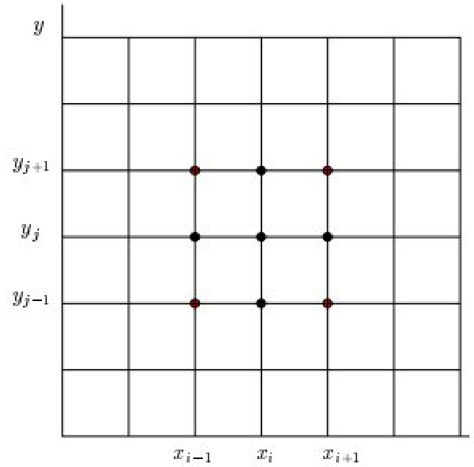
Approximation of mixed derivatives

2D: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x} + \mathcal{O}(\Delta x)^2$$

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} + \mathcal{O}(\Delta y)^2$$

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} + \mathcal{O}(\Delta y)^2$$



Second-order difference approximation

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y} + \mathcal{O}[(\Delta x)^2, (\Delta y)^2]$$

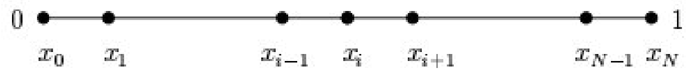
Finite difference method

Example: 1D Poisson equation

Boundary value problem

$$-\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } \Omega = (0,1), \qquad u(0) = u(1) = 0$$

One-dimensional mesh



$$u_i \approx u(x_i), \quad f_i = f(x_i) \qquad x_i = i\Delta x, \quad \Delta x = \frac{1}{N}, \quad i = 0, 1, \ldots, N$$

Central difference approximation $\qquad \mathcal{O}(\Delta x)^2$

$$\begin{cases} -\frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2} = f_i, & \forall i = 1, \ldots, N-1 \\ u_0 = u_N = 0 & \text{Dirichlet boundary conditions} \end{cases}$$

Result: the original PDE is replaced by a linear system for nodal values

Finite difference method

Example: 1D Poisson equation

Linear system for the central difference scheme

$$\begin{cases} i=1 & -\frac{u_0-2u_1+u_2}{(\Delta x)^2} = f_1 \\ i=2 & -\frac{u_1-2u_2+u_3}{(\Delta x)^2} = f_2 \\ i=3 & -\frac{u_2-2u_3+u_4}{(\Delta x)^2} = f_3 \\ & \dots \\ i=N-1 & \frac{u_{N-2}-2u_{N-1}+u_N}{(\Delta x)^2} = f_{N-1} \end{cases}$$

Matrix form $\boxed{Au = F}$ $\quad A \in \mathbb{R}^{N-1\times N-1} \quad u, F \in \mathbb{R}^{N-1}$

$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \dots & \\ & & & -1 & 2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ u_{N-1} \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ f_{N-1} \end{bmatrix}$$

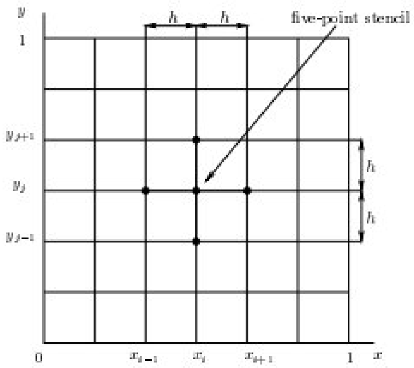
The matrix A is tridiagonal and symmetric positive definite $\Rightarrow$ invertible.

Finite difference method

Example: 2D Poisson equation

Boundary value problem

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f & \text{in } \Omega = (0,1)\times(0,1) \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$



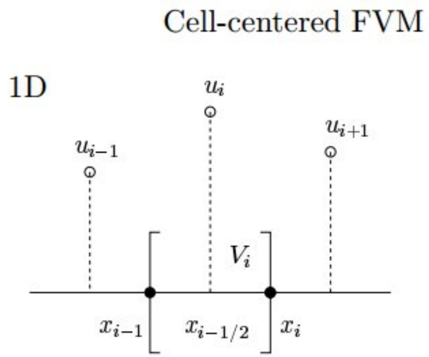
Uniform mesh: $\Delta x = \Delta y = h, \quad N = \frac{1}{h}$

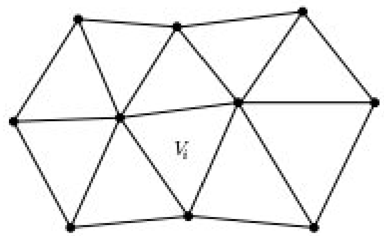
$$u_{i,j} \approx u(x_i, y_j), \quad f_{i,j} = f(x_i, y_j), \qquad (x_i, y_j) = (ih, jh), \quad i, j = 0, 1, \dots, N$$

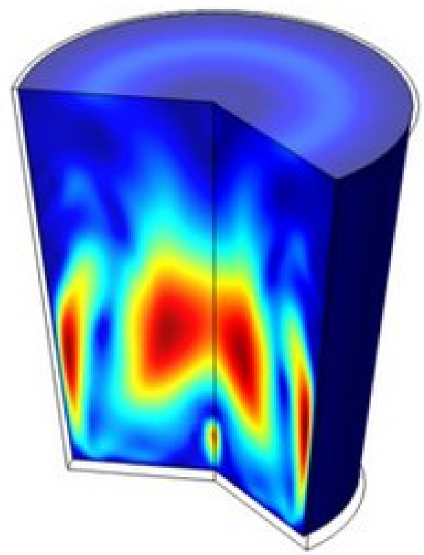
Central difference approximation $\qquad \mathcal{O}(h^2)$

$$\begin{cases} -\frac{u_{i-1,j}+u_{i,j-1}-4u_{i,j}+u_{i+1,j}+u_{i,j+1}}{h^2} = f_{i,j}, & \forall i,j = 1,\dots,N-1 \\ u_{i,0} = u_{i,N} = u_{0,j} = u_{N,j} = 0 & \forall i,j = 0,1,\dots,N \end{cases}$$

Finite volume method







Finite volume method

Finite volume method

The *finite volume method* is based on (I) rather than (D). The integral conservation law is enforced for small control volumes defined by the computational mesh:

$$\bar{V} = \bigcup_{i=1}^{N} \bar{V}_i, \qquad V_i \cap V_j = \emptyset, \quad \forall i \neq j$$

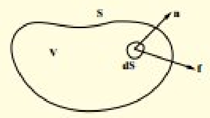
$$u_i = \frac{1}{|V_i|} \int_{V_i} u \, dV \quad \text{mean value}$$

To be specified

- concrete choice of control volumes
- type of approximation inside them
- numerical methods for evaluation of integrals and fluxes

Integral conservation law (I)

$$\frac{\partial}{\partial t} \int_V u \, dV + \int_S \mathbf{f} \cdot \mathbf{n} \, dS = \int_V q \, dV$$



$$\mathbf{f} = \mathbf{v}u - d\nabla u$$

flux function

$$\int_V \frac{\partial u}{\partial t} dV + \int_V \nabla \cdot \mathbf{f} \, dV = \int_V q \, dV$$

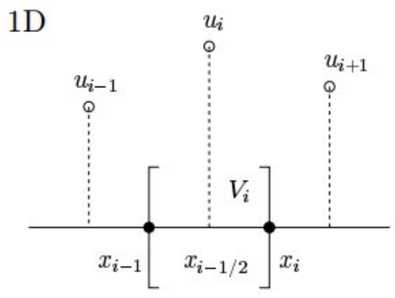
Partial differential equation (D)

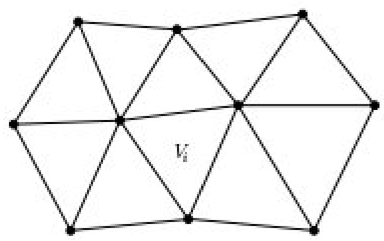
$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f} = q \qquad \text{in } V$$

In steady state $\frac{\partial u}{\partial t} = 0$ so that

$$\nabla \cdot (u\mathbf{v}) = \nabla \cdot (d\nabla u) + q$$

Cell-centered FVM





Finite volume method

Discretization of local subproblems

Integral equation for a single finite volume

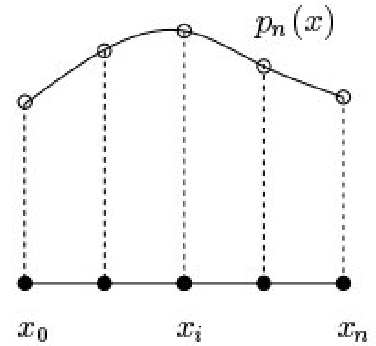
$$\frac{\partial u_i}{\partial t} + \frac{1}{|V_i|} \sum_k \int_{S_k} \mathbf{f} \cdot \mathbf{n}_k \, dS = q_i, \qquad u_i = \frac{1}{|V_i|} \int_{V_i} u \, dV, \quad q_i = \frac{1}{|V_i|} \int_{V_i} q \, dV$$

- the integral conservation law is satisfied for each CV and for the entire domain
- to obtain a linear system, integrals must be expressed in terms of mean values

Numerical integration

$$\int_V f(\mathbf{x}) \, dV \approx \sum_{i=0}^{n} w_i f(\mathbf{x}_i)$$

where $w_i \geq 0$ are the *weights* and $\mathbf{x}_i$ are the *nodes* of the quadrature rule

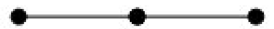


Such formulae can be derived by exact integration of an interpolation polynomial

Finite volume method

Newton-Cotes quadrature rules for intervals

1D

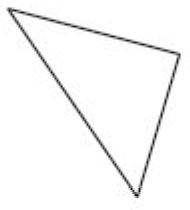


$x_{12} = \frac{x_1+x_2}{2}, \quad V = (x_1, x_2), \quad |V| = x_2 - x_1$

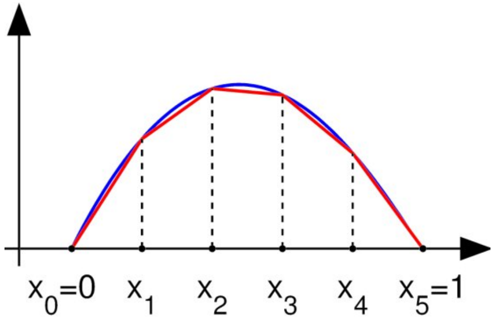
Midpoint rule	$\int_V f(x)\,dV \approx \lvert V\rvert f_{12}$	exact for $f \in P_1(V)$
Trapezoidal rule	$\int_V f(x)\,dV \approx \lvert V\rvert \frac{f_1+f_2}{2}$	exact for $f \in P_1(V)$
Simpson's rule	$\int_V f(x)\,dV \approx \lvert V\rvert \frac{f_1+4f_{12}+f_2}{6}$	exact for $f \in P_3(V)$

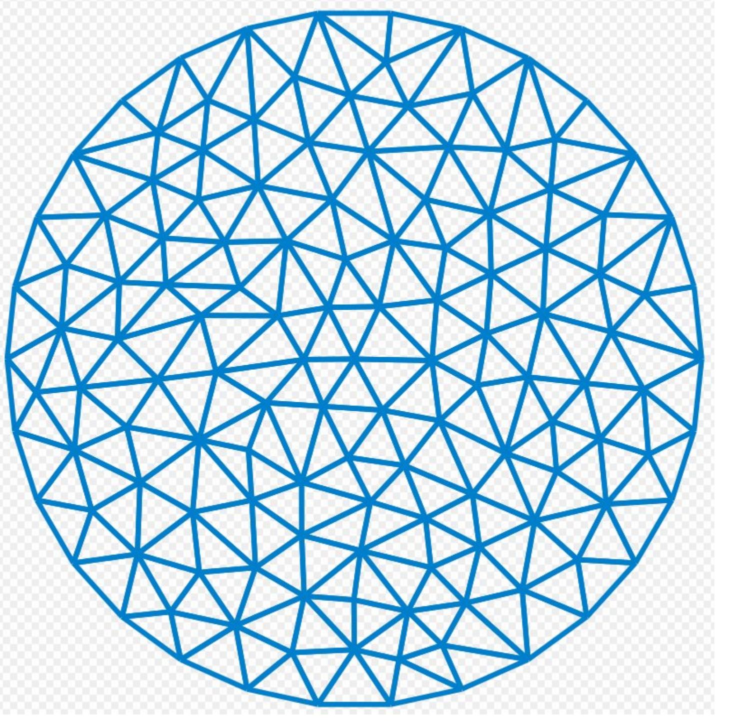
Finite element method

Finite element method



Triangular Finite
Element

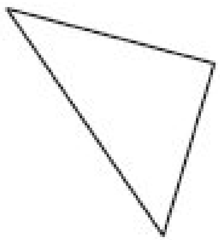




Finite element method

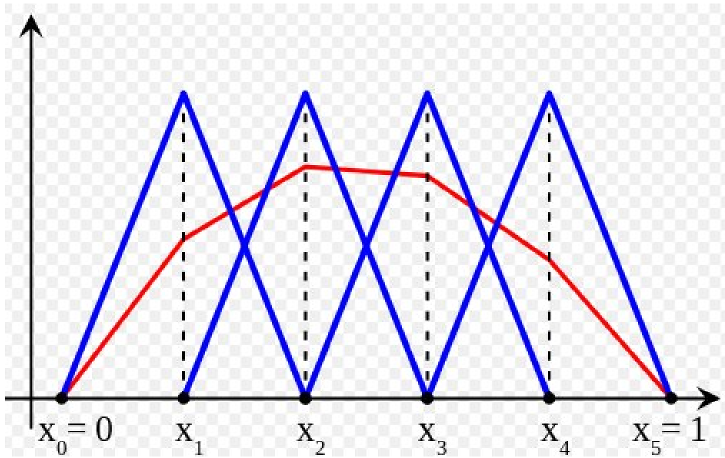
History

Courant (1943) used triangular elements with variational principles to solve vibration problems



Triangular Finite Element

Vikings discovered America



Basis functions *vk* (blue) and a linear combination of them

1960s, the field aroused the interest of many mathematicians

Finite element method

History

1965,NASA develop a general-purpose finite element program NASTRAN

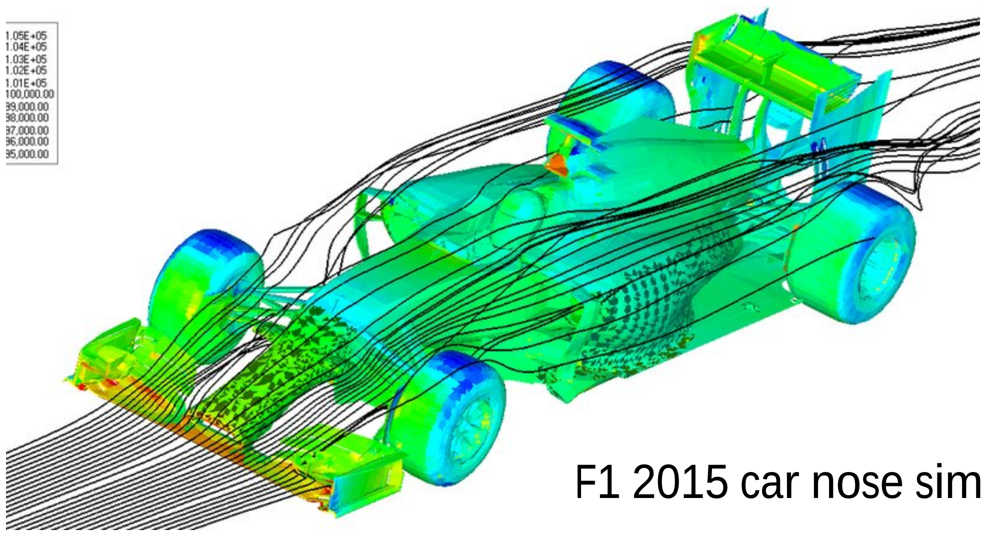
$30 000 000

1969 ANSYS analysis of nuclear reactors

1978 ABAQUS 1989 LS-DYNA

Today: CAD

CFD calculations



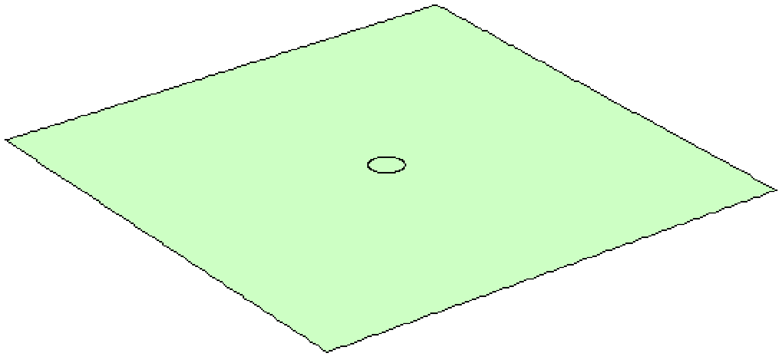
F1 2015 car nose simulation

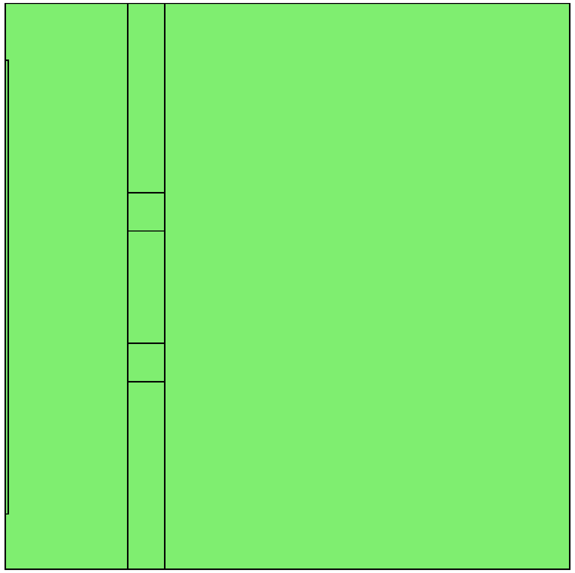
Finite element method

TABLE 16-1: CLASSICAL PDES IN COMPACT AND COMPONENT NOTATION

EQUATION	COMPACT NOTATION	COMPONENT NOTATION (2D)
Laplace's equation	$-\nabla\cdot(\nabla u) = 0$	$-\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = 0$
Poisson's equation	$-\nabla\cdot(c\nabla u) = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Helmholtz equation	$-\nabla\cdot(c\nabla u) + au = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + au = f$
Heat equation	$d_a\frac{\partial u}{\partial t} - \nabla\cdot(c\nabla u) = f$	$d_a\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Wave equation	$e_a\frac{\partial^2 u}{\partial t^2} - \nabla\cdot(c\nabla u) = f$	$e_a\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Convection-diffusion equation	$d_a\frac{\partial u}{\partial t} - \nabla\cdot(c\nabla u) + \beta\cdot\nabla u = f$	$d_a\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + \beta_x\frac{\partial u}{\partial x} + \beta_y\frac{\partial u}{\partial y} = f$

Finite element examples

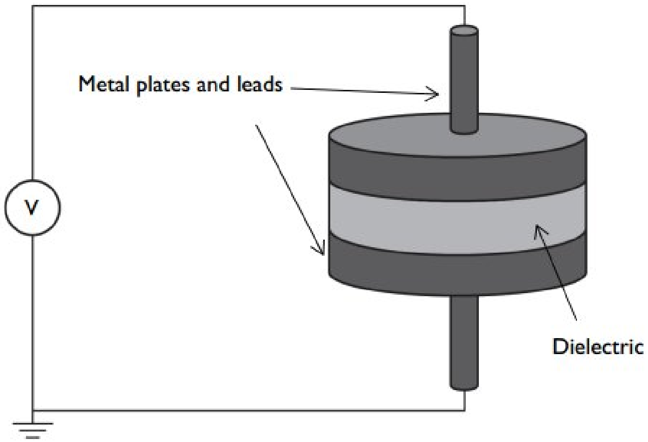




Finite element examples

Poisson's equation $-\nabla \cdot (c\nabla u) = f$

$$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$$



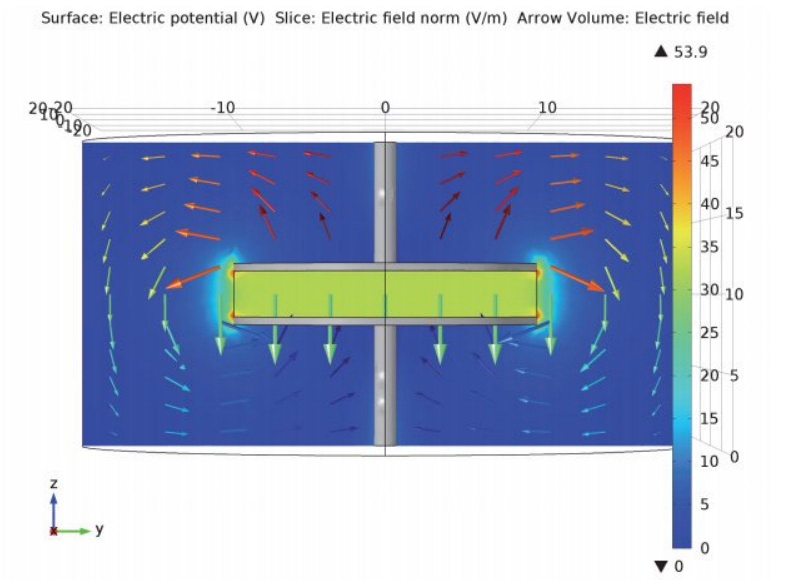
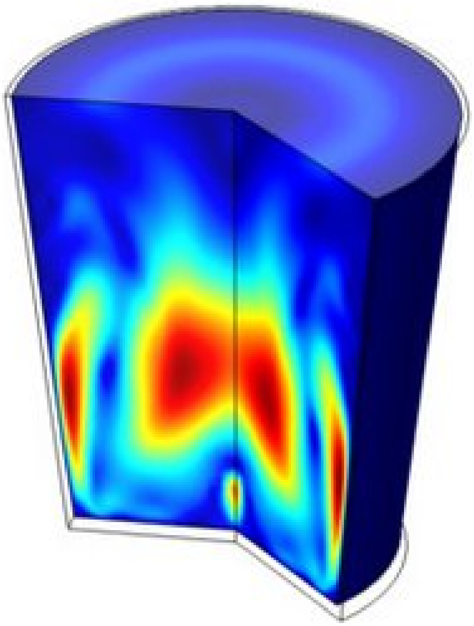


Figure 2: The electric field strength in the dielectric and air domain surrounding the capacitor.

Finite element examples

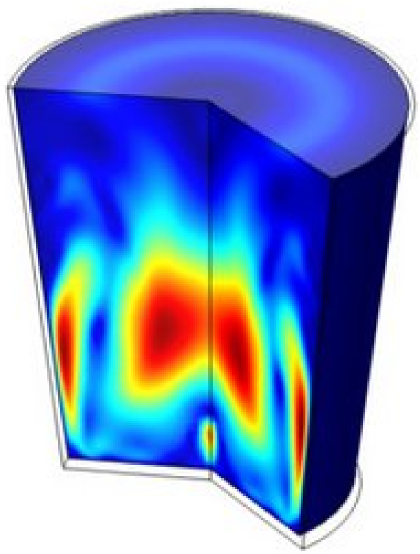
Heat equation	$d_a\frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u) = f$	$d_a\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$

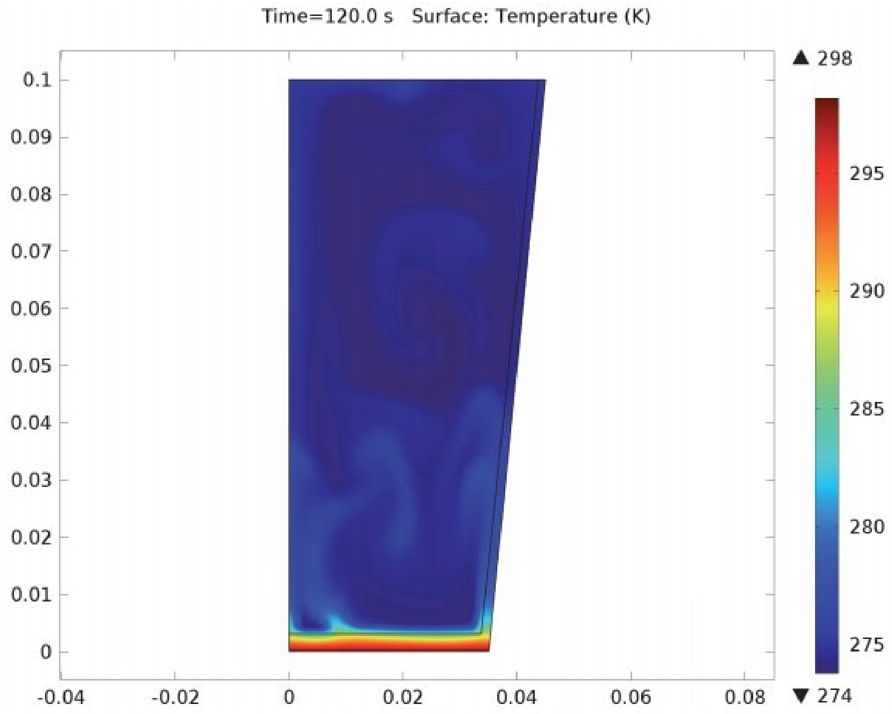


Finite element examples

Convection-diffusion equation	$d_a\frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u) + \beta \cdot \nabla u = f$	$d_a\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + \beta_x\frac{\partial u}{\partial x} + \beta_y\frac{\partial u}{\partial y} = f$

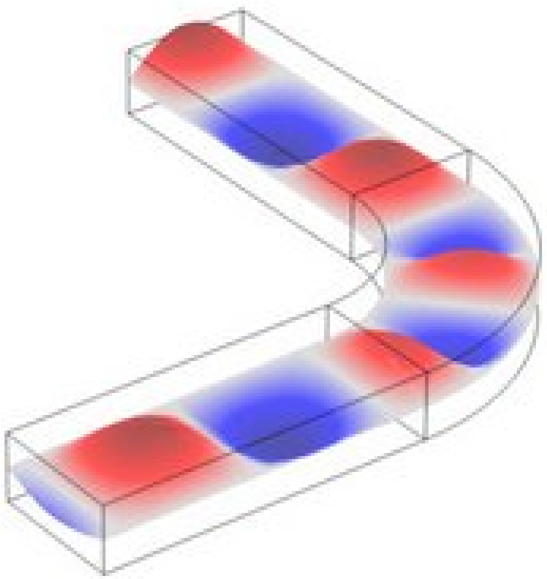
water





Finite element examples

Helmholtz equation	$-\nabla \cdot (c\nabla u) + au = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + au = f$



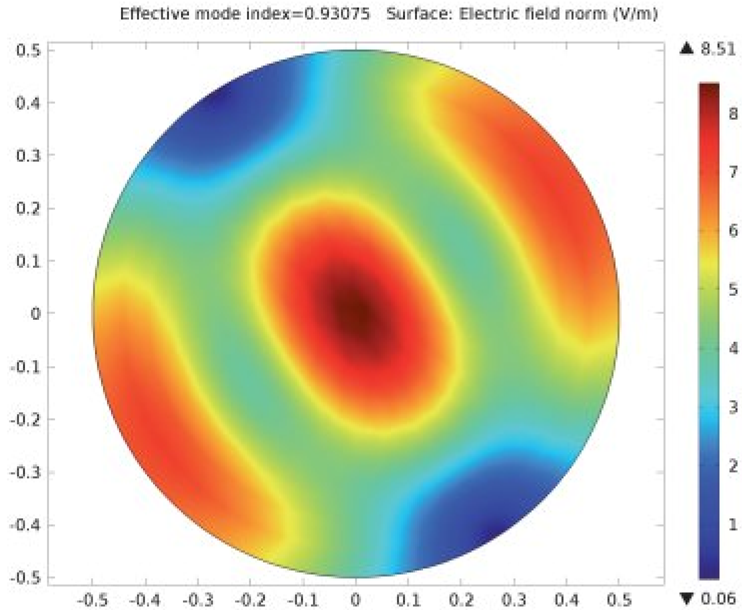
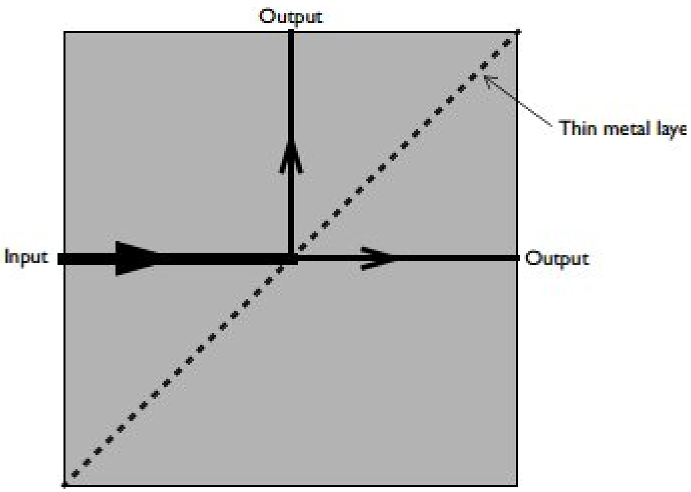


Figure 1: The surface plot visualizes the norm of the electric field for the effective mode index 0.9308 - $2.208 \cdot 10^{-6}j$.

Finite element examples

Wave equation	$e_a \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c \nabla u) = f$	$e_a \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c \frac{\partial u}{\partial y}\right) = f$



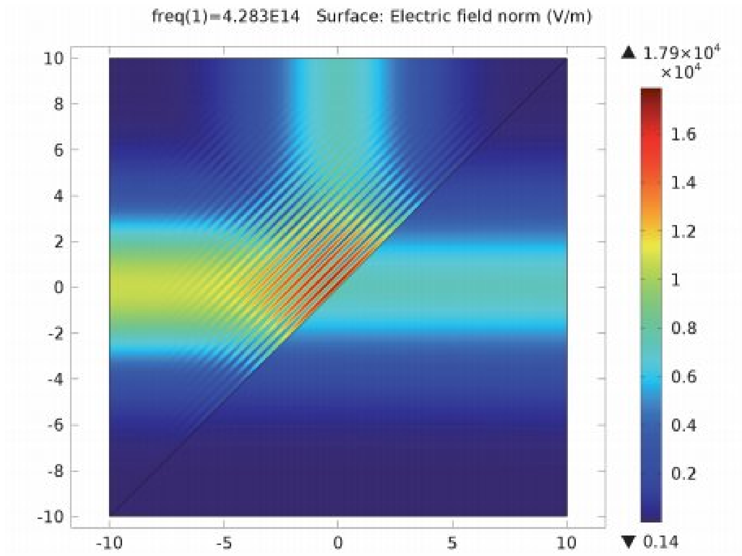


Figure 2: The electric field intensity shows that the incoming beam is split into two beams of approximately equal intensity.