1D topology and 1D topological insulators (v2)

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In these notes, I try to translate the content of Chapter 1 (the one-dimensional SSH model) of our book (Asboth et al., A Short Course on Topological Insulators) to the language of topology, an established branch of mathematics. I think the treatment in our book is very useful, because it exposes the key ideas with simple physical pictures, without the complexity of formal mathematics. Nevertheless, I believe it is very useful to familiarize with the underlying mathematical concepts and relations, at least for the reasons that (1) it hopefully enables you to efficiently discuss with, and learn from, mathematicians about this subject, and (2) to provide a framework for understanding topology in condensed-matter systems in more generality, beyond the specific examples treated in the course.

In particular, I plan to connect the SSH model to basic concepts of topology such as *continuity*, *homeomorphism*, *chart*, *atlas*, *transition map*, *manifold*, (*smooth manifold* and *orientable manifold*), *homotopy*, *degree*, and the *complete homotopy invariant*.

After my 45-minute lecture, which is roughly covered in this write-up, Gergő Pintér will give another 45-minute lecture, mostly focusing on the extension of the above topology concepts for two-dimensional lattice models, in relation to the Qi-Wu-Zhang model.

I. CONTINUITY AND HOMEOMORPHISM

1. A key concept in topology is a *continuous function*. Roughly speaking, a continuous function is a function for which small changes in the input result in small changes in the output. Let's introduce continuity via an example: the Brillouin zone of the 1D SSH model.

2. Naively, based on band-structure diagrams seen throughout this course, we'd think that the Brillouin zone is the $]-\pi,\pi]$ interval. However, we also remember that $k=\pi$ is equivalent to $k=-\pi$. But that means that actually the Brillouin zone is the unit circle.

3. Mathematicians denote the unit circle as S^1. (Sometimes they call it the one-dimensional torus, and denote it as T^1.) Throughout this lecture, we will identify the unit circle either with the unit circle in the real plane $\mathbb{R}^2$ (the set $\{(x,y)\in\mathbb{R}^2,\, x^2+y^2=1\}$), or with the unit circle in the complex plane $\mathbb{C}$ (the set $\{u\in\mathbb{C},\, |u|^2=1\}$).

4. So we see that the $]-\pi,\pi]$ interval is quite similar to S^1, but they are not the same. In fact, we can parametrize S^1 with $]-\pi,\pi]$ in a continuous fashion:

$$\varphi_0 :]-\pi,\pi] \to S^1, \alpha \mapsto \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{1}$$

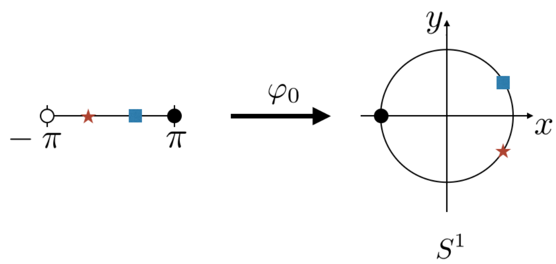

FIG. 1. The continuous bijective parametrization of the unit circle, see Eq. (1).

5. This parametrization φ_0 is a continuous function: if you pick any point $\alpha \in]-\pi, \pi]$, and approach this point with some α', then the value $\varphi_0(\alpha')$ will approach $\varphi_0(\alpha)$.

6. The parametrization φ_0 is a one-to-one function, i.e., a bijection, and therefore it has an inverse, φ_0^{-1}. Writing out explicitly,

$$\varphi_0^{-1} : S^1 \to]-\pi, \pi], \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \arg(x + iy), \tag{2}$$

where $\arg : \mathbb{C} \to]-\pi, \pi]$ outputs the phase of a complex number.

7. Interestingly, φ_0^{-1} is not continuous, even though it is the inverse of a continuous function. The point $-\boldsymbol{e}_x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is a point that is mapped to π, but if $\boldsymbol{r} \in S^1$ approaches $-\boldsymbol{e}_x$ from below the x axis, then the value of φ_0^{-1} at $\boldsymbol{r}$ approaches $-\pi$ instead of π.

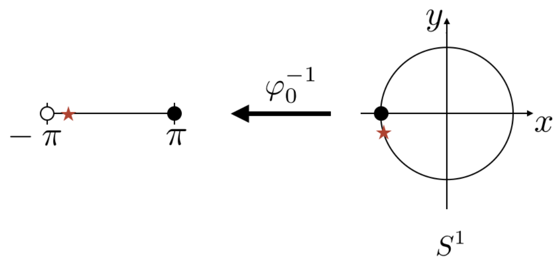

FIG. 2. The inverse of the continuous bijective map φ_0 is not continuous, hence φ_0 is not a homeomorphism.

8. Another key concept in topology is *homeomorphism*, which is a name for functions that are continuous, bijective, and their inverse is also continuous. We have seen that φ_0 above is not a homeomorphism: although it is continuous and invertible, its inverse is not continuous. But it is easy enough to use that counterexample to come up with an example: just leave out π from the domain of φ_0:

$$\varphi_0' :]\pi, \pi[\to S^1 \setminus \{-\boldsymbol{e}_x\},\ \alpha \mapsto \varphi_0(\alpha). \tag{3}$$

This is now a homeomorphism.

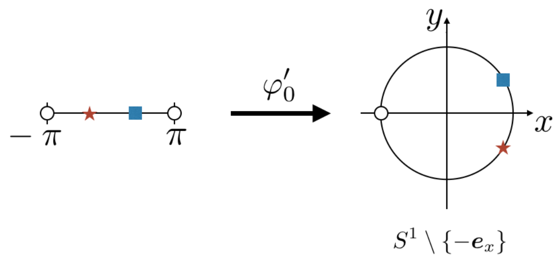

FIG. 3. A continuous bijective map with a continuous inverse: such a map is called a homeomorphism.

II. MANIFOLD

1. Another key concept in topology is the *manifold*. What is it, what are the simplest examples, and how are they related to topological insulators?

2. In fact, the Brillouin-zone of a 1D lattice model, which we have identified above with the unit circle S^1, is a one-dimensional manifold: it locally resembles the one-dimensional Euclidean space $\mathbb{R}$ near each of its points. More precisely, each of its point has a neighborhood that can be parametrized with an open interval of $\mathbb{R}$ through a homeomorphism.

3. For example, take the point $\boldsymbol{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of the unit circle. Consider the neighborhood U of this point that contains the whole circle except the opposite point $-\boldsymbol{e}_x$:

$$U = S^1 \setminus \{-\boldsymbol{e}_x\} \tag{4}$$

Luckily enough, φ_0' defined above is just a parametrization of U on an open interval of $\mathbb{R}$, and we have also seen that φ_0' is a homeomorphism.

Can we find similar neighborhoods and parametrizations for all points of S^1? Yes. On the one hand, U and φ_0' are just fine for any points except $-\boldsymbol{e}_x$. On the other hand, for $-\boldsymbol{e}_x$, one option is to take its neighborhood $S^1 \setminus \{\boldsymbol{e}_x\}$, and use the parametrization

$$\varphi_\pi' :]0, 2\pi[\to S^1 \setminus \{\boldsymbol{e}_x\},\ \alpha \mapsto \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{5}$$

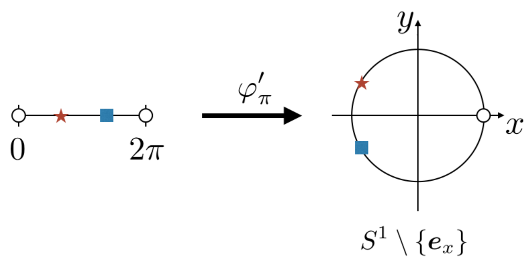

FIG. 4. A second homeomorphism (besides φ'_0) serving as a local homeomorphic parametrization of the unit circle S^1.

4. Up to now, we have seen that S^1 is a one-dimensional manifold, which means that you can locally parametrize it with $\mathbb{R}$ everywhere, despite the fact that you cannot parametrize it globally with $\mathbb{R}$.

5. The local, homeomorphic $\mathbb{R} \to S^1$ parametrizations discussed above (φ'_0 and φ'_π) will be called *charts*. (They are also called *coordinate charts*, *coordinate patches*, *coordinate maps*, or *local frames*.)

6. A collection of charts that cover the whole manifold is called an *atlas*. For the unit circle S^1, the two charts φ'_0 and φ'_π together constitute an atlas.

7. Given two charts, their images might have an intersection. For example, the intersection of the images of φ'_0 and φ'_π is $S^1 \setminus \{\boldsymbol{e}_x, -\boldsymbol{e}_x\}$. Taking an open interval in the pre-image of this intersection by φ'_0, we can map this interval into the pre-image of this intersection by the other chart φ'_π. Such a map is called a *transition map*.

   For example, in the figure below we illustrate this transition map:

$$\tau :]\pi/3, 2\pi/3[\to]\pi/3, 2\pi/3[, \tau = (\varphi'_\pi)^{-1} \circ \varphi'_0. \tag{6}$$

   (Note that the domain and range of a transition map happen to be the same here, incidentally. Also incidentally, this τ happens to be the identity function.)

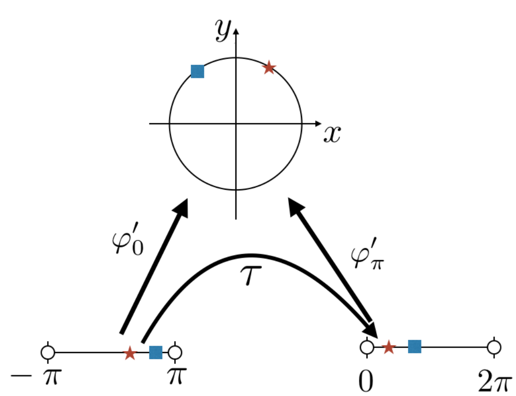

FIG. 5. A transition map between two charts of the unit circle S^1.

8. In the context of topological insulators, we consider *smooth manifolds*: these are manifolds whose transition maps are smooth, that is, are differentiable as many times as needed. Note that for transition maps, it makes sense to talk about their differentiability, since they are $\mathbb{R} \to \mathbb{R}$ functions.

9. The concept of a one-dimensional manifold is naturally generalized to higher dimensions. One example of a two-dimensional manifold is the Brillouin zone of the Qi-Wu-Zhang model (or other 2D lattice models), where the Brillouin zone is actually a (two-dimensional) torus, denoted by T^2.

10. **Exercise:** Write out explicitly an atlas of a torus. Think of this torus as a 2D surface embedded in 3D space.

11. **Exercise:** Which of these letters are manifolds in the sense defined above: d, D, o, O, x, X, y, Y, 8?

III. ORIENTABLE MANIFOLDS

1. Take the above-defined smooth manifold: the unit circle S^1 with the atlas consisting of charts φ'_0 and φ'_π.

2. The transition maps are $\mathbb{R} \to \mathbb{R}$ maps, and they are also bijective (since they are composed of bijective maps). This implies that for any transition map τ, its derivative $\dot{\tau} : \mathbb{R} \to \mathbb{R}$ has a definite sign, either positive, or negative. If the derivative of a transition map is positive, then it is called an *orientation-preserving* transition map.

3. In our example atlas of S^1, any transition map is orientation-preserving. For example, the transition map τ defined in Eq. (6) has a constant positive derivative: $\dot{\tau} = 1$.

4. If a smooth manifold can be described by an orientation-preserving atlas, then it is called an orientable smooth manifold. So, the unit circle S^1 is an orientable manifold.

5. Orientability can be naturally extended to larger-dimensional manifolds. For n-dimensional manifolds, the transition maps are $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n, \boldsymbol{r} \mapsto \boldsymbol{f}(\boldsymbol{r})$ maps, and therefore the 'derivative' is not a number, but a matrix: the Jacobian: $J_{ij}(\boldsymbol{r}) = (\partial_i f_j)(\boldsymbol{r})$. Bijectivity of the transition map implies that the sign of the determinant of the Jacobian of any transition map is either positive or negative in the whole domain of the transition map. Orientation-preserving maps are those whose Jacobian determinant is positive. An n-dimensional orientable manifold is one which can be described with an atlas whose transition maps are all orientation-preserving.

6. **Exercise:** Which manifolds are orientable? Sphere (usually denoted as S^2, and meant to be the surface of a sphere), torus, cylinder, Mobius strip.

IV. AN INSULATOR'S VALENCE BAND IS A MAP BETWEEN TWO MANIFOLDS

1. Consider the example of the one-dimensional SSH model, whose bulk momentum-space Hamiltonian reads:

$$H_{\text{bulk}}(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix}. \tag{7}$$

   In generalized versions of this model, the function in the off-diagonal matrix element of the Hamiltonian can involve higher harmonics as well.

2. For each wave number k, we can associate a valence-band Bloch spinor $|u_1(k)\rangle$, which is a normalized lower-energy eigenstate of the matrix $H_{\text{bulk}}(k)$. Furthermore, $|u_1(k)\rangle$ is *balanced*, meaning that its two components have the same weight; this is a consequence of the chiral symmetry of the Hamiltonian.

3. We can think of this association as a function from the Brillouin zone S^1 to the balanced unit vectors of $\mathbb{C}^2$:

$$u_1 : S^1 \to \{\text{balanced unit vectors of } \mathbb{C}^2\} \tag{8}$$

4. Such a Bloch spinor is not necessarily continuous. For example, consider the fully dimerized limit $v = 1$ and $w = 0$, when the 'energy eigenstates do not depend on k', but we can still define a strange, non-continuous Bloch spinor u_1 that maps to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from some subset of the wave numbers and maps to $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ from the rest of the wave numbers.

5. To make sure that an SSH insulator defines a continuous function, we substitute the Bloch spinor function with the Bloch projector function: for any Bloch spinor u_1, we can define

$$P_1 : S^1 \to \{\text{balanced 1D projectors of } \mathbb{C}^2\}, k \mapsto |u_1(k)\rangle \langle u_1(k)| . \tag{9}$$

We can also think about this function as a map from the Brillouin zone and the space of 1D subspaces of $\mathbb{C}^2$,

$$P_1 : S^1 \to \{\text{balanced 1D subspaces of } \mathbb{C}^2\}. \tag{10}$$

We call this map P_1 the *valence band* of the SSH model.

6. Physicists know the Bloch sphere, and like to think about the balanced 1D subspaces of $\mathbb{C}^2$ as the equator of the Bloch sphere, which is just the unit circle of the two-dimensional xy plane. The corresponding homeomorphism is

$$\boldsymbol{p} : \{\text{balanced 1D subspaces of } \mathbb{C}^2\} \to S^1, \rho \mapsto \begin{pmatrix} \mathrm{Tr}(\rho\sigma_x) \\ \mathrm{Tr}(\rho\sigma_y) \end{pmatrix}, \tag{11}$$

From now on, we will use the map $P = \boldsymbol{p} \circ P_1$ to describe the valence band. Clearly,

$$P : S^1 \to S^1, \tag{12}$$

hence, indeed, the valence band is a map between two (orientable and smooth) manifolds.

V. HOMOTOPY CLASSES OF SSH VALENCE BANDS ARE CHARACTERIZED BY THEIR DEGREE

1. In our case, we can consider the valence bands P and Q of two different SSH models. Both P and Q are $S^1 \to S^1$ maps. They are called *homotopic*, if they can be continuously deformed into each other.

2. Can they? Not necessarily. For example, the 'trivial fully dimerized limit' $v = 1$, $w = 0$ and the 'topological fully dimerized limit' $v = 0$, $w = 1$, cannot be deformed into each other, hence they are not homotopic.

3. The concept of homotopy can be formalized as follows. Two maps $f, g : X \to Y$ are homotopic, if there is an *interpolating map* or *connecting map* or *deformation map* $H : X \times [0, 1] \to Y$ that is continuous and $f = H(., 0)$ and $g = H(., 1)$.

   In the context of topological insulators, when we say that two models are 'adiabatically equivalent' often we mean that the corresponding valence band maps are homotopic.

4. The SSH valence bands are maps between one-dimensional manifolds. Maps between same-dimensional manifolds can be characterized by their *degree*. Roughly speaking, the degree $\deg(P)$ of the map P is essentially the sign-corrected sum of the number of coincidences of an arbitrary point of the target space and the image of P. (Essentially the same concept is explained in our book, in Figure 1.5.)

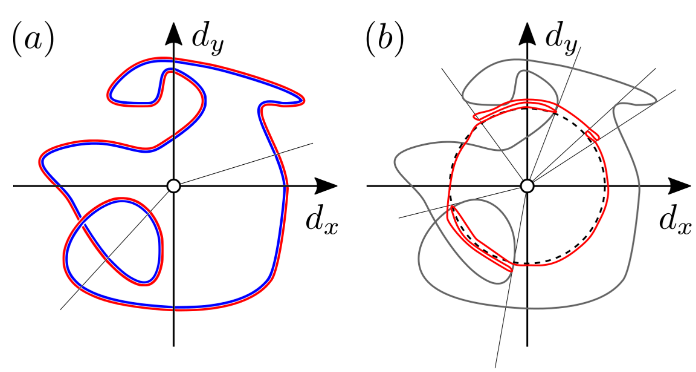

FIG. 6. Figure 1.5 from the book, (b) showing an illustration of an $S^1 \to S^1$ map.

More formally, in the one-dimensional case, for $P : S^1 \to S^1$ in particular, the degree is defined as follows:

(a) Can you find a point in the target space which lies outside the image of P? If you can, then the degree is 0.

(b) If you cannot find such a point in the target space, then pick an arbitrary 'regular' point $\boldsymbol{n}$ of the target space.

(c) The point $\boldsymbol{n}$ will have at least one, but possibly more pre-image points in the source space; denote them with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, etc.

(d) Take sufficiently small charts φ_1, φ_2, φ_3, etc, for all source-space points $\boldsymbol{e}_1$, etc., and sufficiently small chart χ for $\boldsymbol{n}$, such that each $\mathbb{R} \to \mathbb{R}$ map $\tau_i = \chi^{-1} \circ P \circ \varphi_i$ is bijective. If this cannot be done, then the chosen point $\boldsymbol{n}$ is not regular enough; choose a different one and redo the procedure.

(e) Denote the pre-image of the source-space point $\boldsymbol{e}_1$ by φ_1 as x_1, etc. Calculate the sign s_1 of the derivative $\dot{\tau}_1(x_1)$, etc, and sum up those signs to obtain the degree of P: $\deg(P) = s_1 + s_2 + \dots$.

Note that the result of this procedure is independent of the choice of the target-space point $\boldsymbol{n}$.

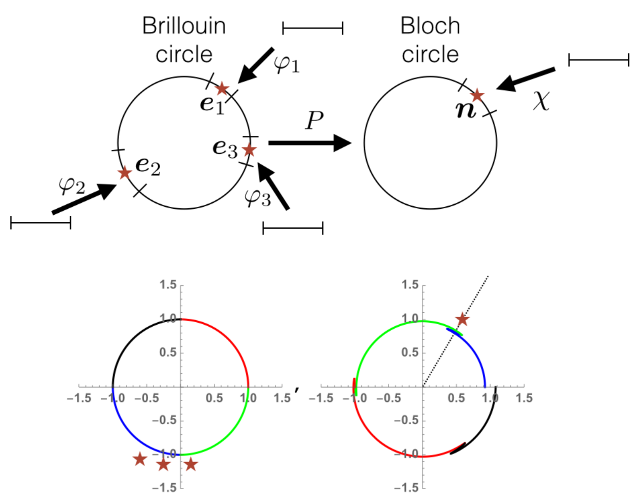

FIG. 7. Illustration of the definition of the degree of a $P : S^1 \to S^1$ map. The colored circles illustrate the image of the map $P : e^{ik} \mapsto -e^{ik} - 0.75e^{-2ik}$: the left circle shows the domain, the right circle shows the map itself.

5. Proposition: the degree is a complete homotopy invariant, that is, two $S^1 \to S^1$ maps are homotopic if and only if they have the same degree.

6. Remark: the degree can be generalized to two-dimensional manifolds (or higher) using the Jacobian.

7. Remark: valence bands of Rice-Mele models are $S^1 \to S^2$ maps. They are all trivial.

8. **Exercise:** There is the famous donut-mug homotopy, see, e.g., at `https://en.wikipedia.org/wiki/Homotopy`. What is the source space and what is the target space in that case?