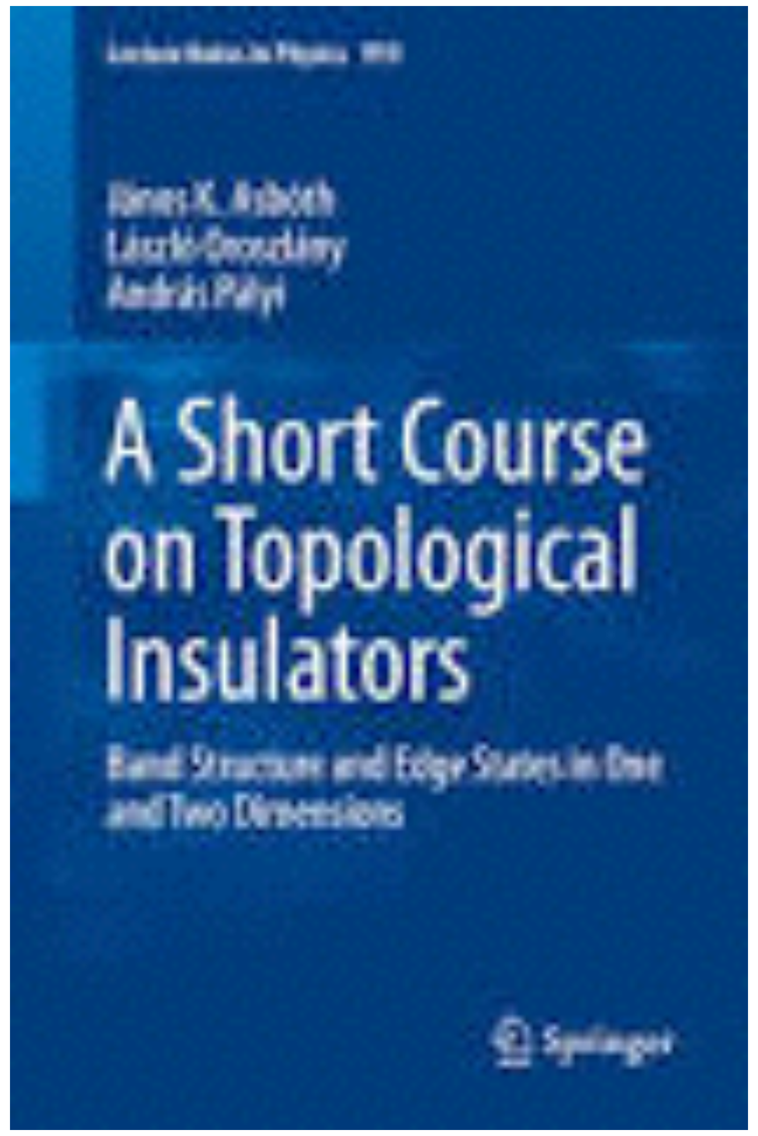
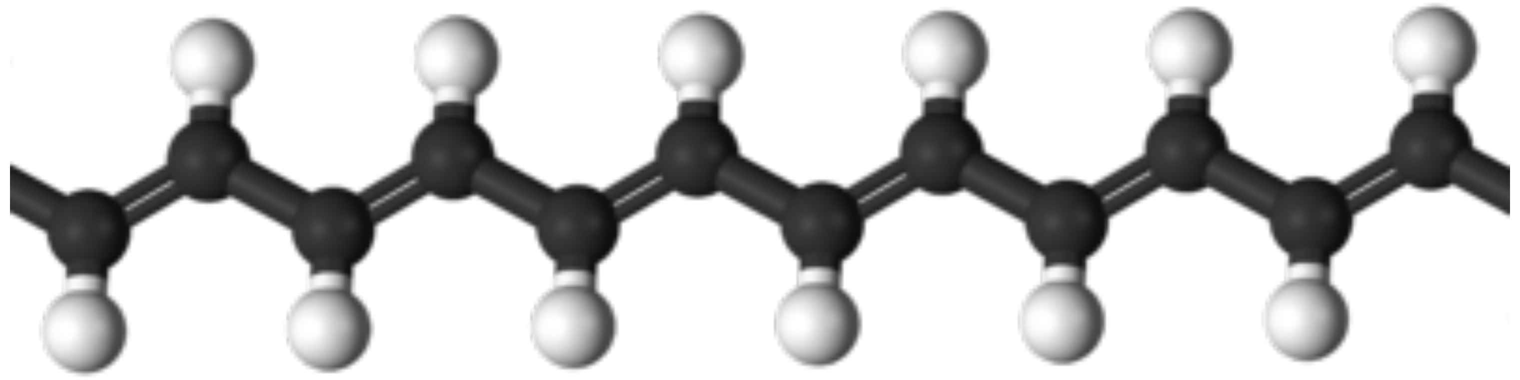
Chapter 1: The Su-Schrieffer-Heeger (SSH) model

Topological Insulators, 2021. 09. 20.

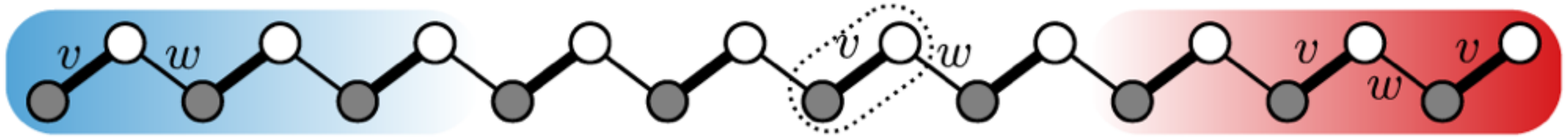
Asboth et al.
arXiv:1509.02295



Polyacetylene



SSH model



Bulk-boundary correspondence

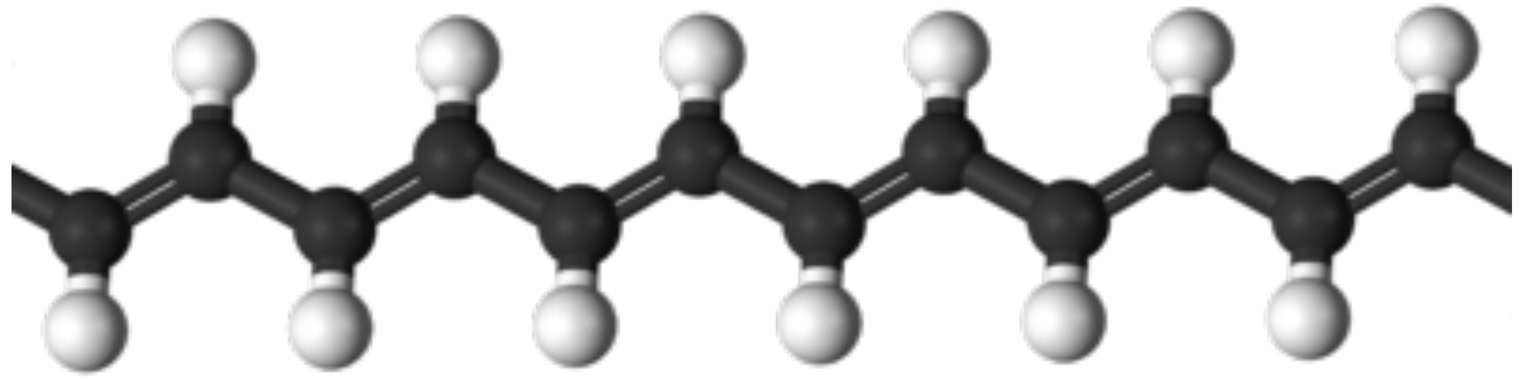
If the bulk has nontrivial topology,
then the edge has disorder-resistant bound states

(Chapter 1 shows the simplest model where this can be demonstrated.)

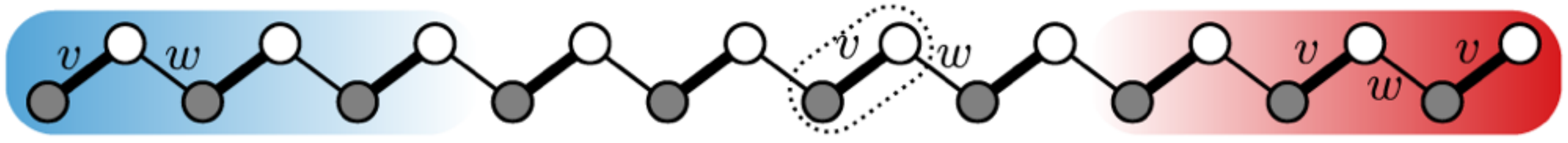
SSH is a tight-binding toy model for polyacetylene

Polyacetylene

Su et al., PRL 1979



Su-Schrieffer-Heeger (SSH) model of polyacetylene



lattice constant a

half filling: 1 electron per atom

Real-space tight-binding SSH Hamiltonian:

$$\hat{H} = v \sum_{m=1}^{N} \left(|m,B\rangle \langle m,A| + h.c. \right) + w \sum_{m=1}^{N-1} \left(|m+1,A\rangle \langle m,B| + h.c. \right).$$

intracell hopping

intercell hopping

For N=4:

$$H = \begin{pmatrix} 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \end{pmatrix}$$

k-space Hamiltonian maps unit circle to complex plane

Real-space tight-binding SSH Hamiltonian:

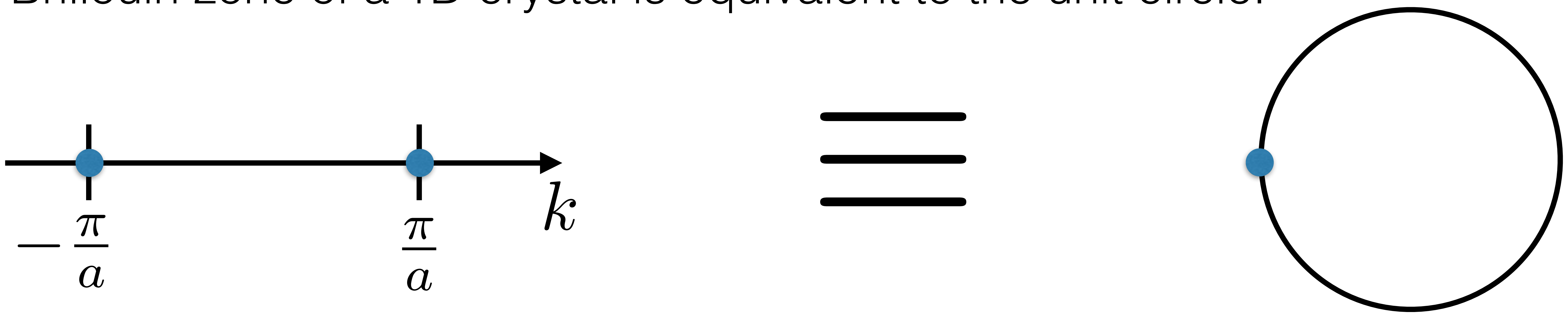
$$\hat{H} = v \sum_{m=1}^{N} \left(|m,B\rangle \langle m,A| + h.c. \right) + w \sum_{m=1}^{N-1} \left(|m+1,A\rangle \langle m,B| + h.c. \right).$$

k-space SSH Hamiltonian:

$$\boldsymbol{d}(k) = \begin{pmatrix} v + w\cos k \\ w \sin k \end{pmatrix}$$

$$H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{-ik} & 0 \end{pmatrix} = \boldsymbol{d}(k) \cdot \boldsymbol{\sigma}$$

Brillouin zone of a 1D crystal is equivalent to the unit circle:

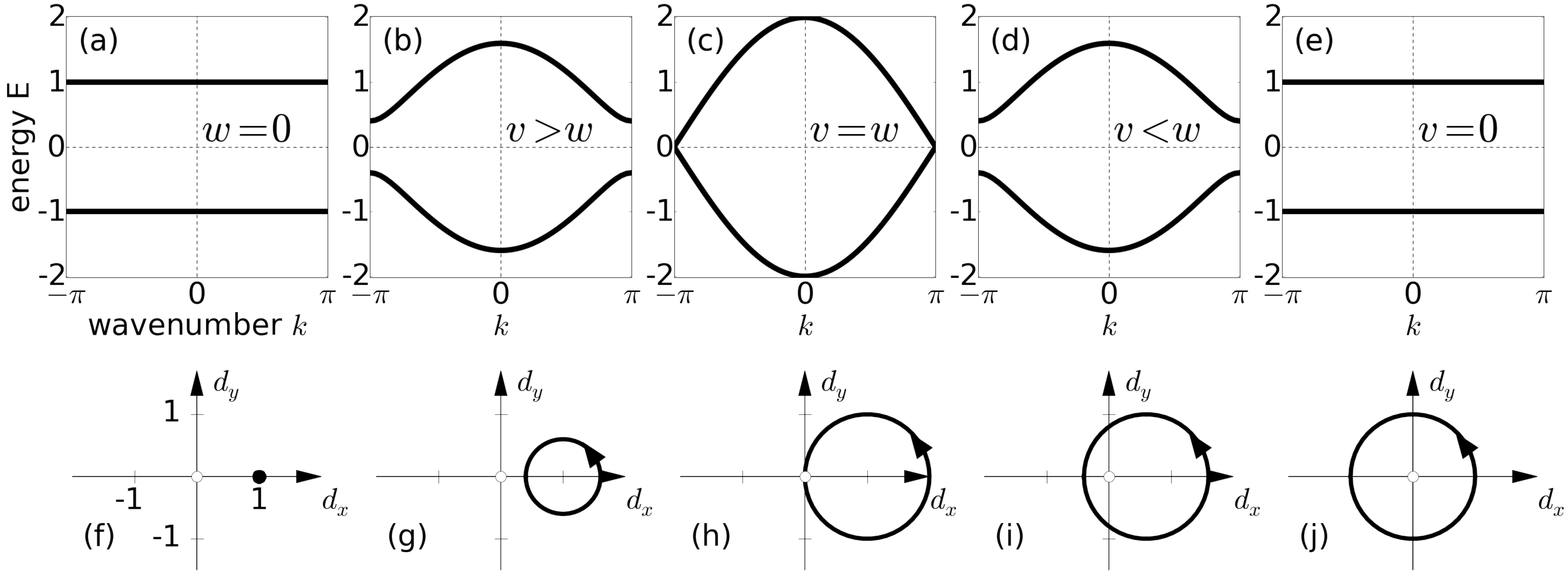


$$f_{v,w} : \text{unit circle} \to \mathbb{C}, k \mapsto v + we^{-ik}$$

An insulating SSH Hamiltonian has a topological invariant

k-space SSH Hamiltonian:

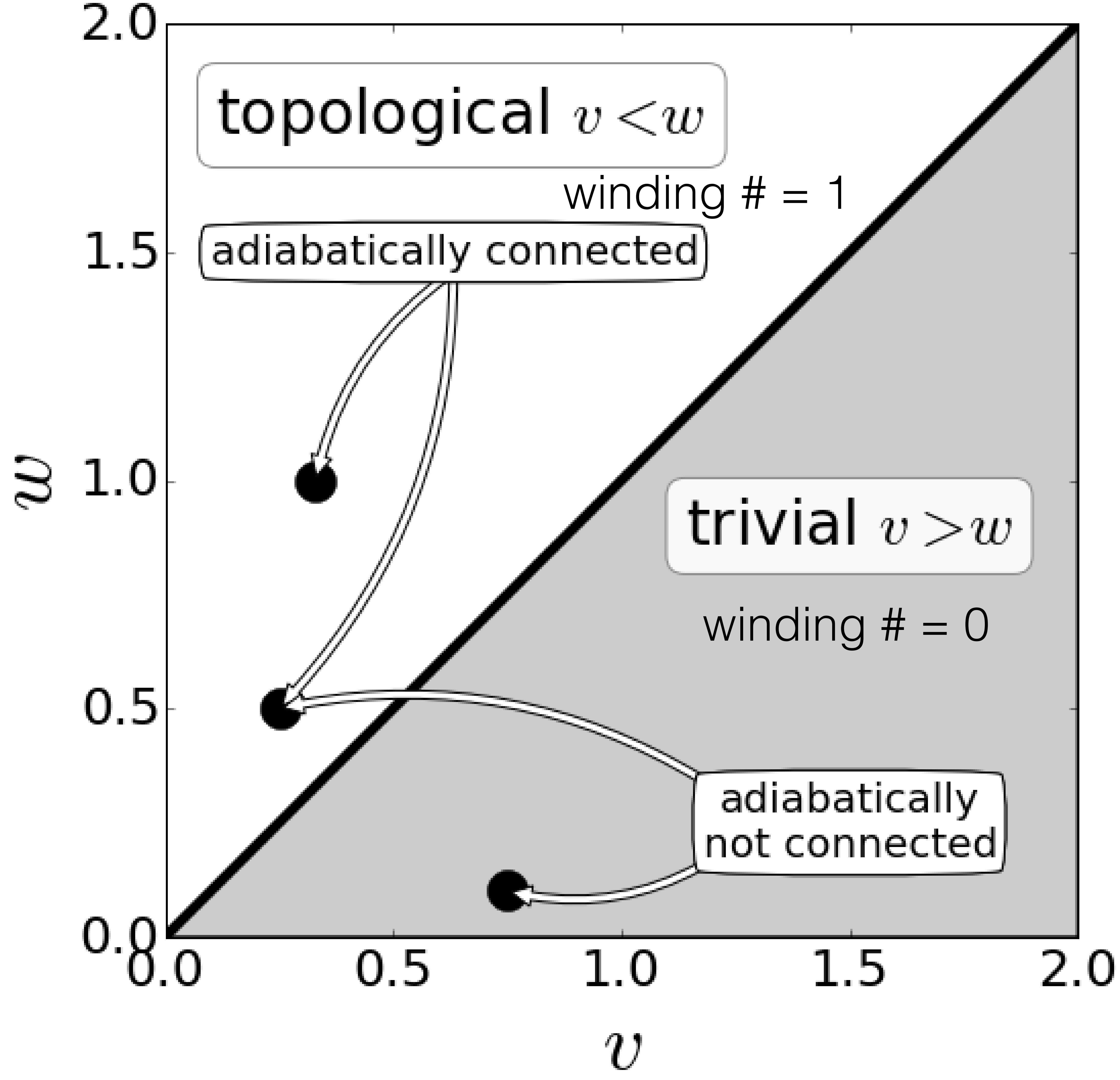
$$H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix}$$



insulator	insulator	metal	insulator	insulator

winding number → 0   0   1   1

SSH parameter space has two topological phases

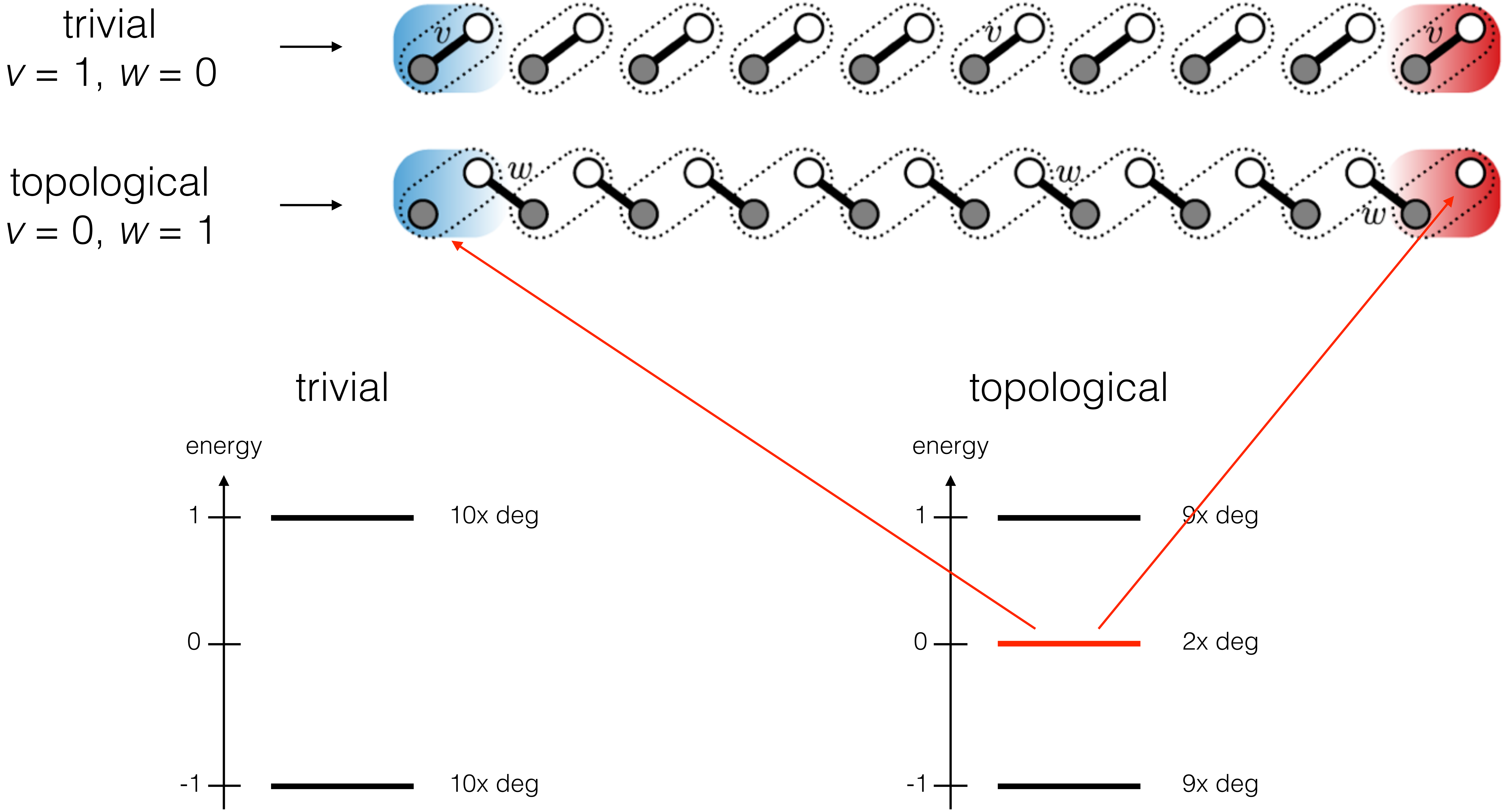


Bulk-boundary correspondence

If the bulk has nontrivial topology,
then the edge has disorder-resistant bound states

Zero intracell hopping implies zero-energy states at edges

Fully dimerized limits of the SSH Hamiltonian:



SSH Hamiltonians have chiral symmetry

Definition: A unitary and hermitian operator Γ is a *chiral symmetry* of the Hamiltonian H if $\Gamma H \Gamma^\dagger = -H$.

Remark: For electrons in crystals, we usually consider local chiral symmetry.

SSH Hamiltonians have chiral symmetry:

For example, N = 4:

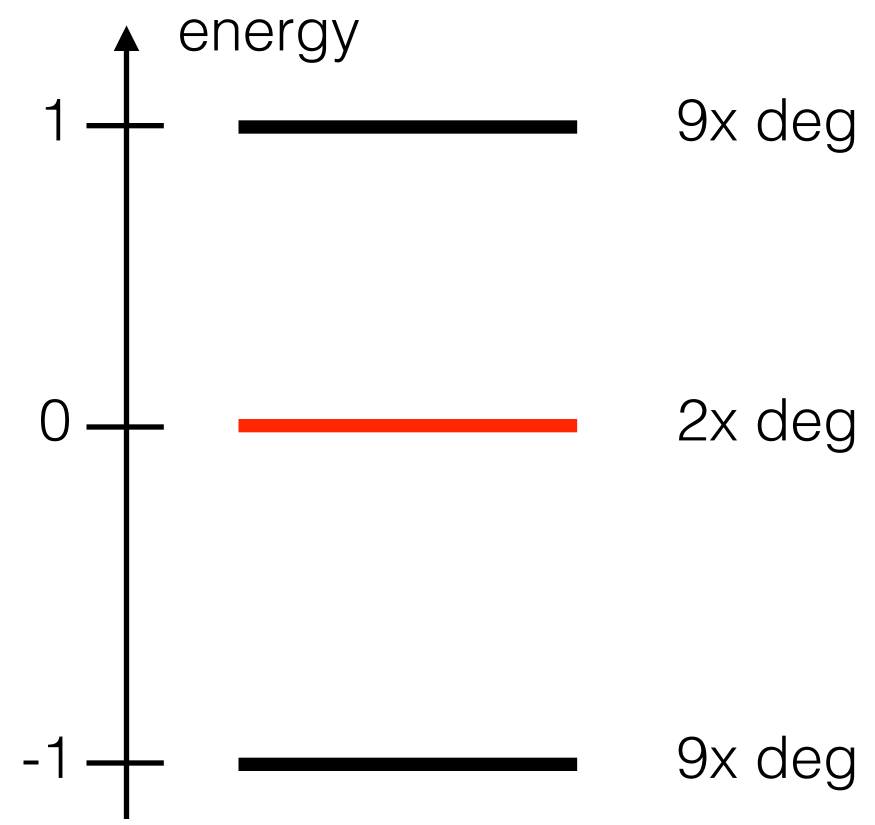
$$H = \begin{pmatrix} 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Consequences of chiral symmetry

Up-down symmetric energy spectrum: $H\psi = E\psi$ implies $H(\Gamma\psi) = -E(\Gamma\psi)$

Finite-energy eigenstates have a 'chiral partner' at opposite energy

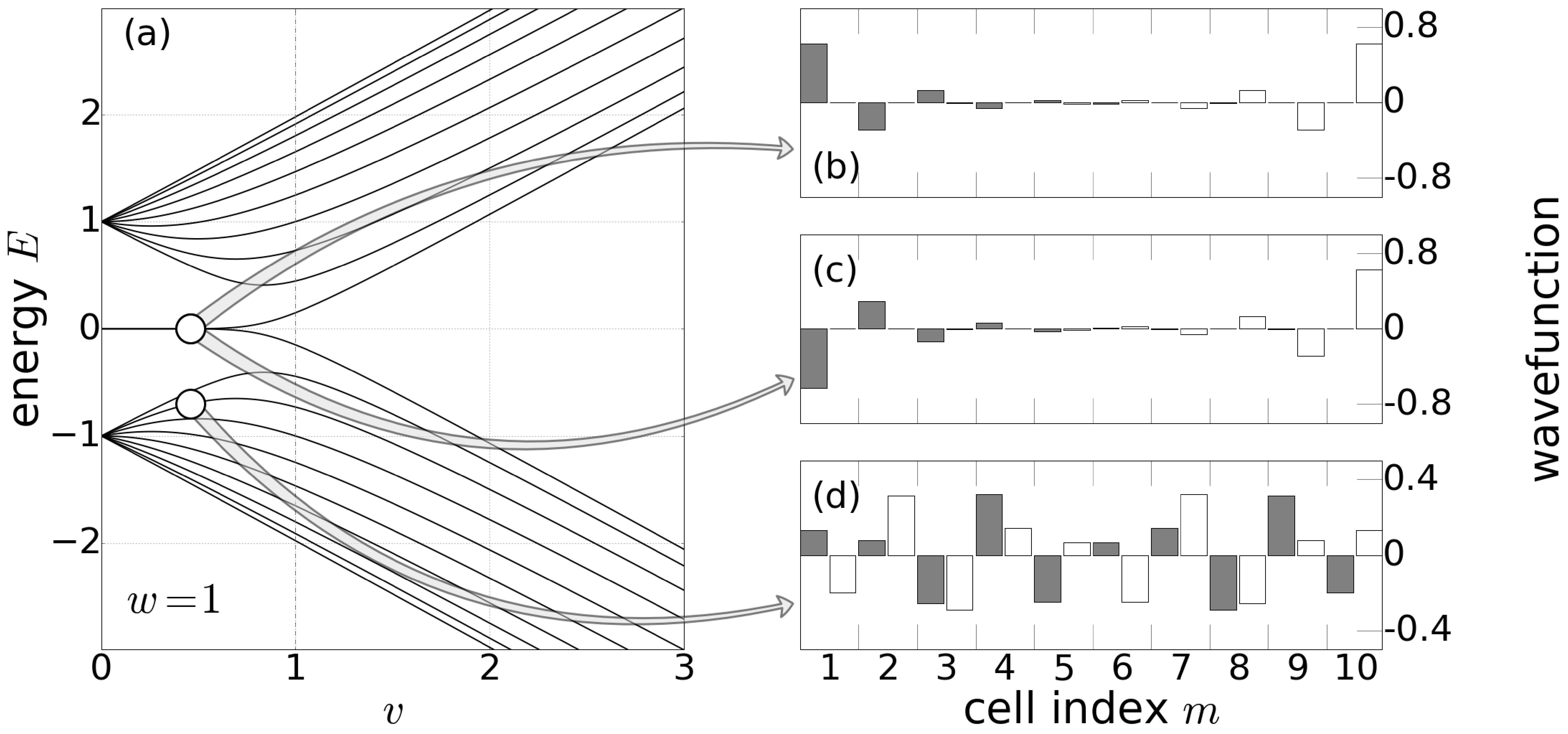
A zero-energy eigenstate might be its own chiral partner



Example: fully dimerized topological SSH chain

Chiral symmetry implies edge states in topological SSH

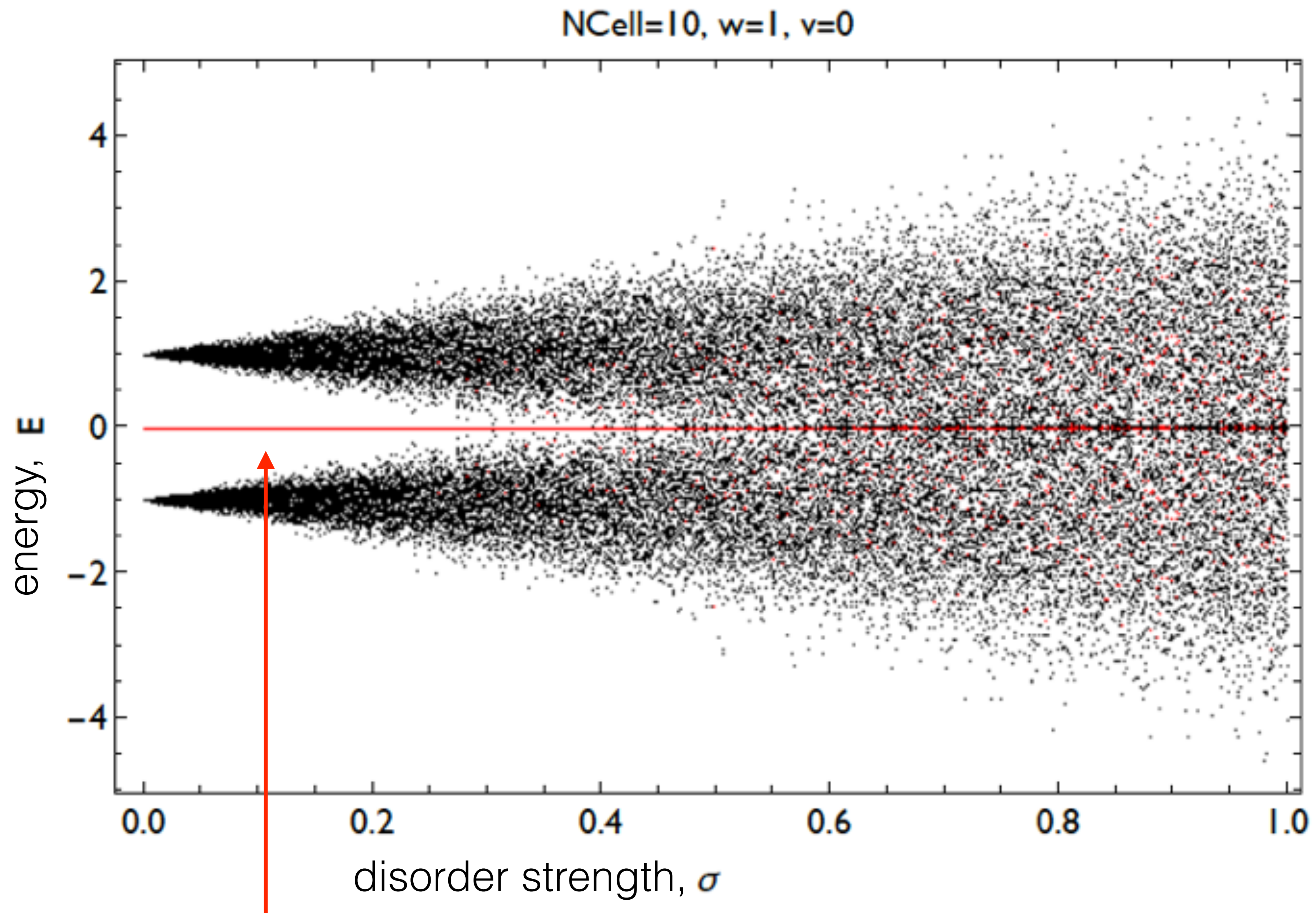
Take long fully dimerized topological SSH chain (v = 0, w = 1).
Switch on a uniform intercell hopping v.
Does the zero-energy edge state survive?
It does: its energy sticks to zero due to chiral symmetry.
The energy can leave zero only if the left and right edge states hybridize.



bulk-boundary correspondence

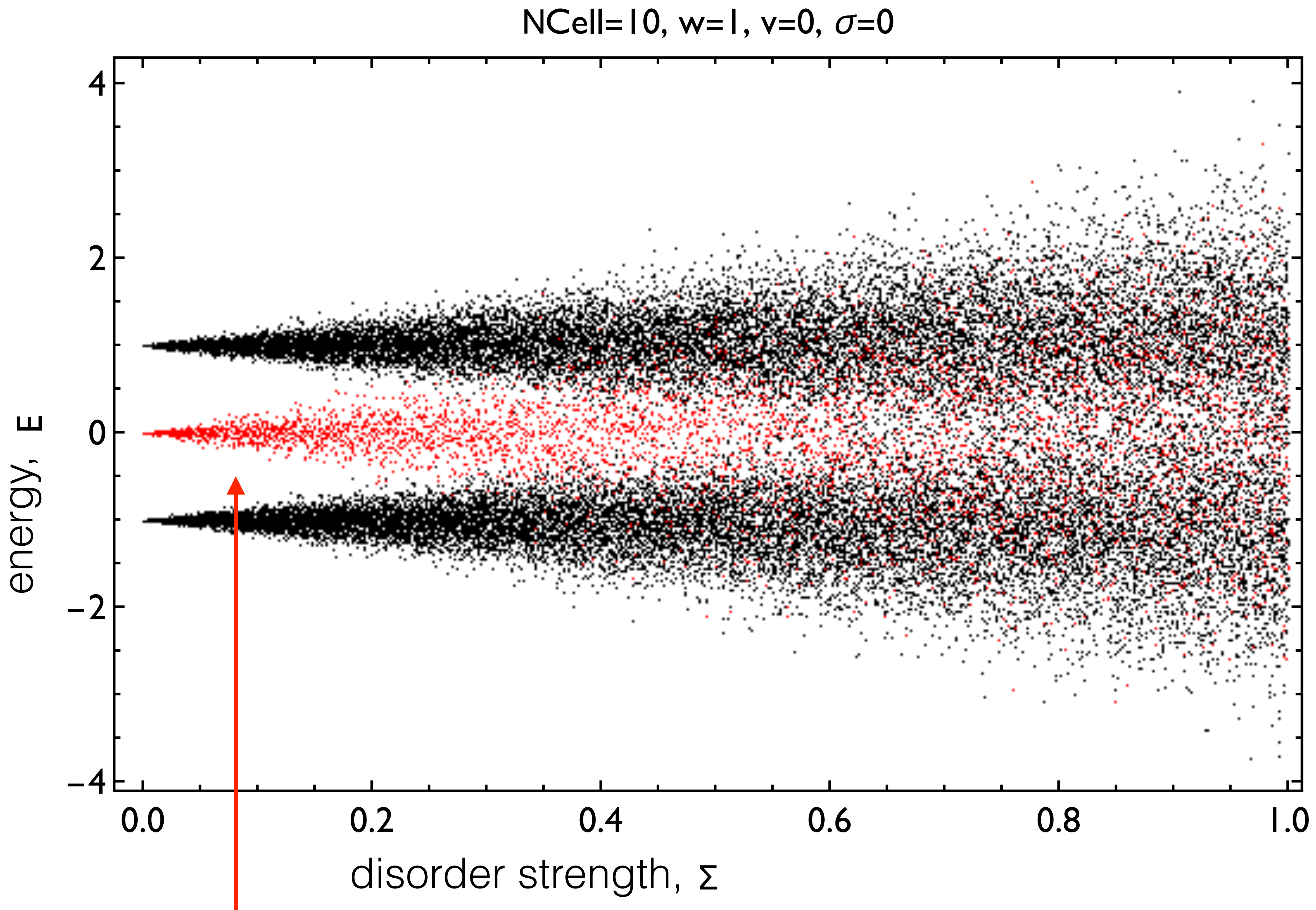
Edge states are robust against chiral-symmetric disorder

Hopping disorder
(respects chiral symmetry)



zero-energy edge states
survive disorder

On-site disorder
(breaks chiral symmetry)



zero-energy edge states:
energy gets scattered by disorder

images from Péter Boross

SSH is one creature in the zoo of topological insulators

spatial dimensions

symmetry classes

Cartan\\d	1	2	3	
Complex case:				
A	0	$\mathbb{Z}$	0	quantum Hall & quantum anomalous Hall effects
AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	
Real case:				
AI	0	0	0	
BDI	$\mathbb{Z}$	0	0	SSH model
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0	
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	quantum spin Hall effect
CII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	
C	0	$2\mathbb{Z}$	0	
CI	0	0	$2\mathbb{Z}$	

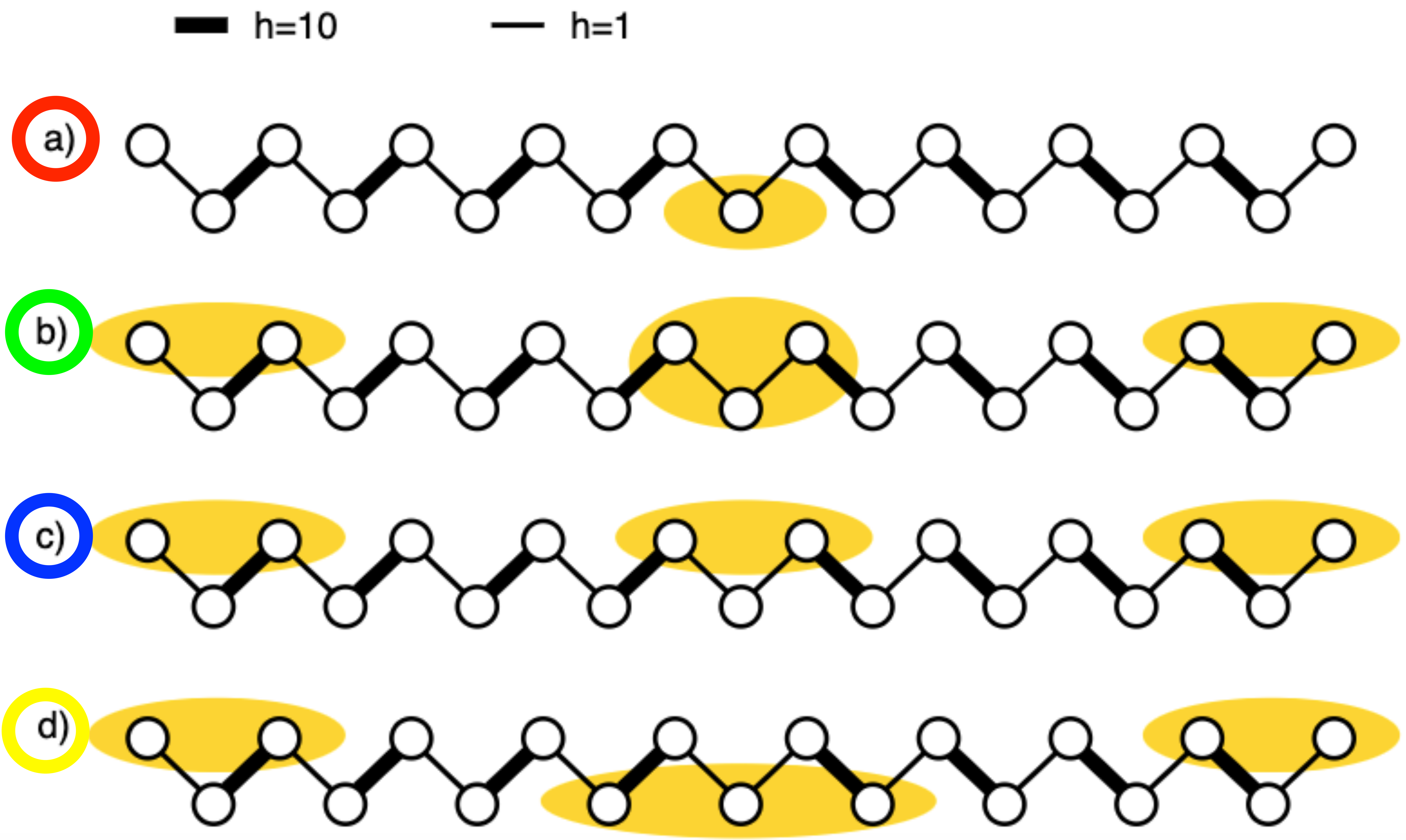
Table from A. W. W. Ludwig, Physica Scripta (2016)

Bulk-boundary correspondence

If the bulk has nontrivial topology,
then the edge has disorder-resistant bound states

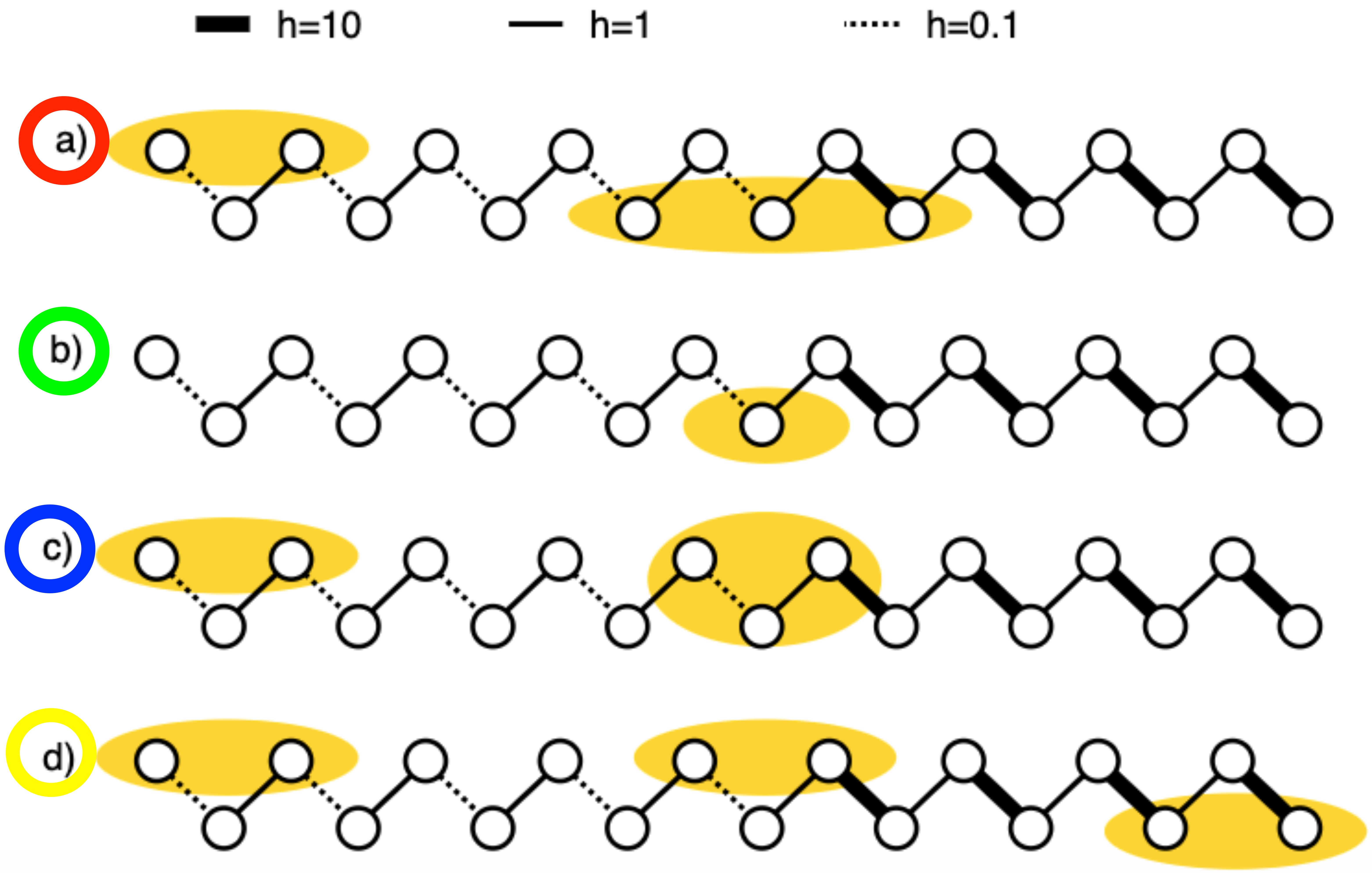
Where are the edge states?

An SSH chain is depicted below, with 2 types of hopping amplitudes h (different line styles). Where are the edge states? The shading indicates sites
over which the edge state wavefunctions extend. Please choose the best answer.



Where are the edge states? 1

An SSH chain is depicted below, with 3 types of hopping amplitudes h (different line styles). Where are the zero-energy bound states? The shading indicates sites over which the wavefunctions extend. Please choose the best answer.



SSH glued

We take two SSH chains with open ends. We glue them into a circle. As a result, the total number of zero-energy bound states...

a) cannot change

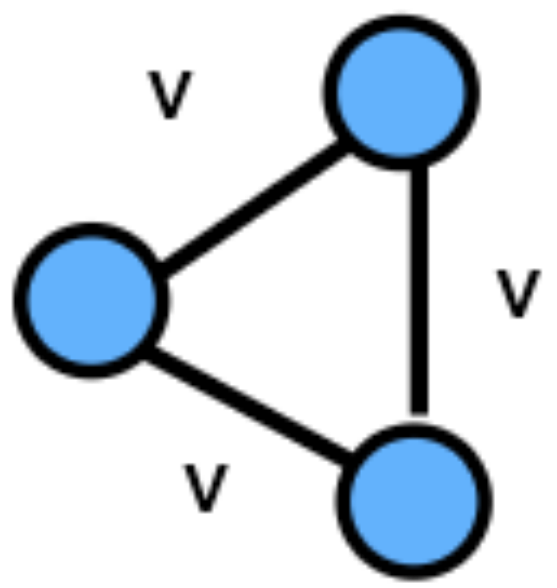
b) can increase

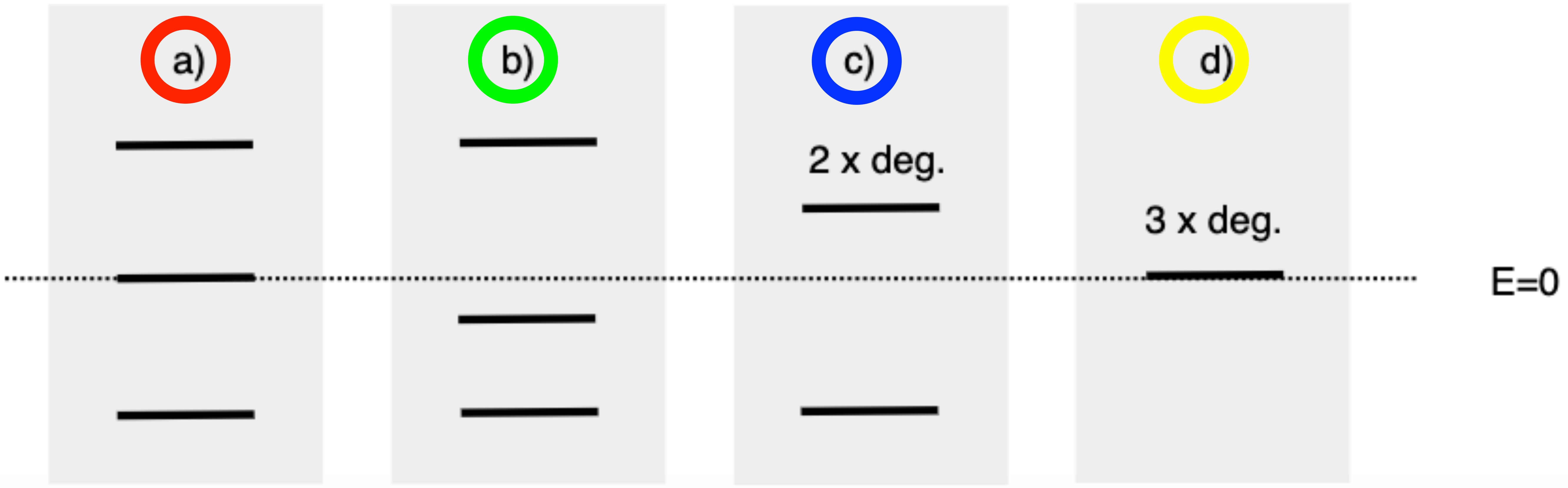
c) can decrease

d) can increase or decrease

3-site ring

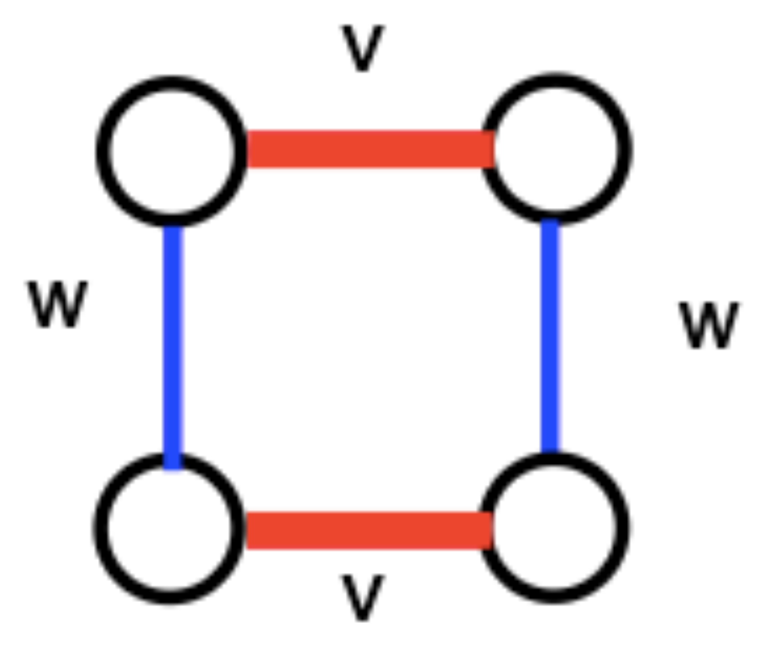
Which is the spectrum of a 3-site ring?
No onsite potentials, just real valued nearest neighbor hopping.
Dotted line = zero of energy



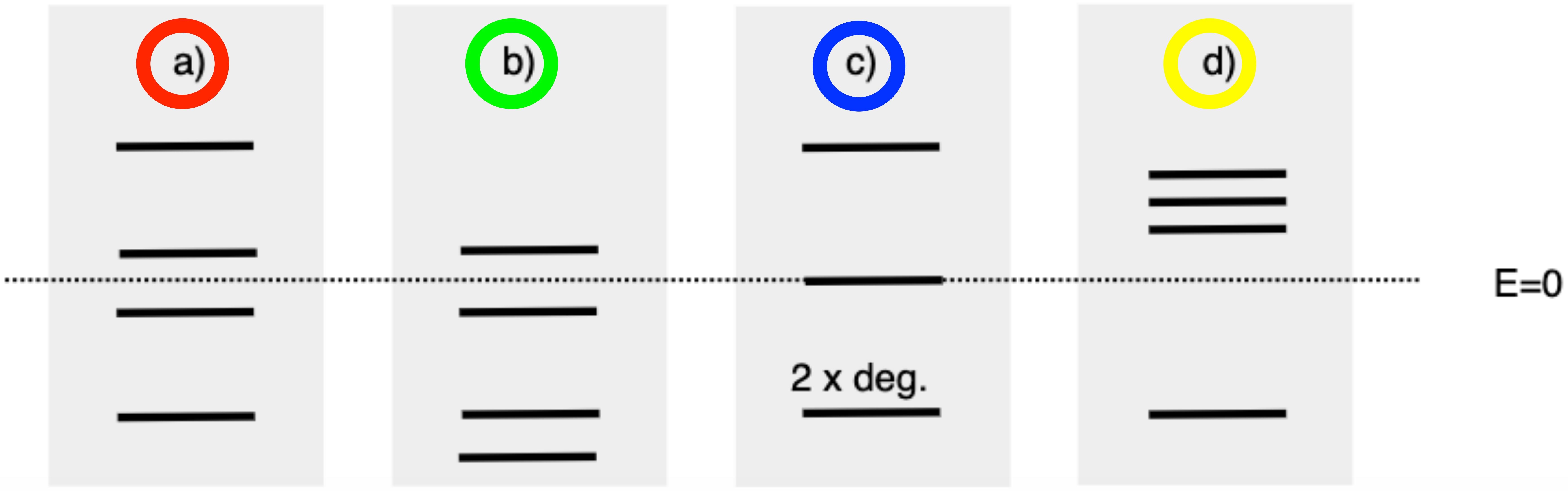


4-site ring 1.

Which is the spectrum of the 4-site ring?
No onsite potential, just nearest neighbor hopping.
Dotted line = zero of energy.

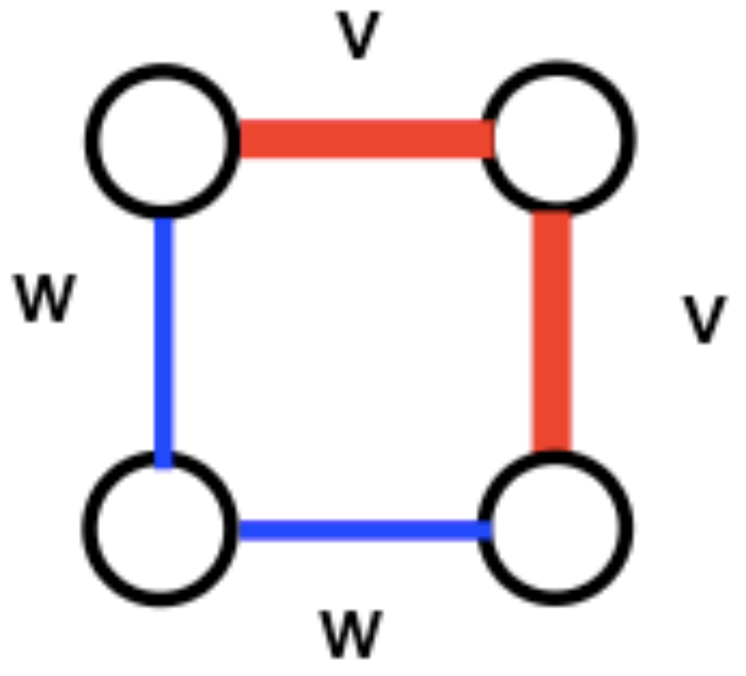


$v \neq w \qquad v, w > 0$

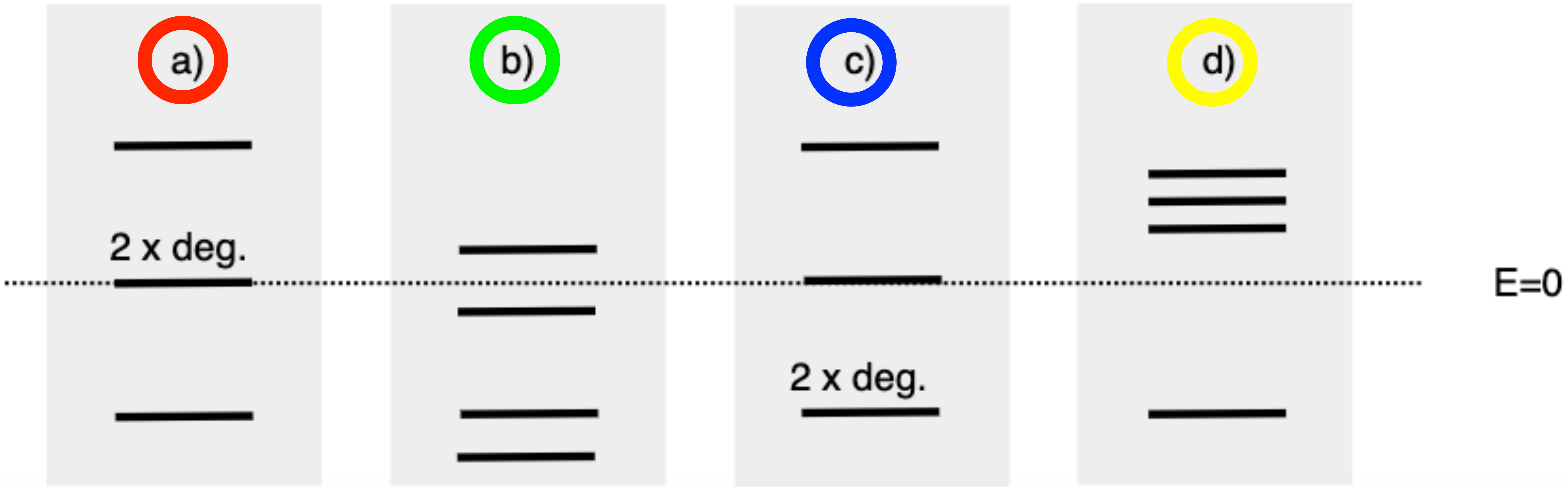


4-site ring 2.

Which is the spectrum of the 4-site ring?
No onsite potential, just nearest neighbor hopping.
Dotted line = zero of energy.



$$v \neq w \qquad v, w > 0$$



Zero modes and chiral symmetry 1.

For zero-energy eigenstates of a single-particle Hamiltonian with chiral symmetry, it is true that…

a) they are chiral symmetric partners of themselves.

b) they break chiral symmetry.

c) their total number is always odd.

d) they can be chiral symmetric partners of themselves.

Zero modes and chiral symmetry 2.

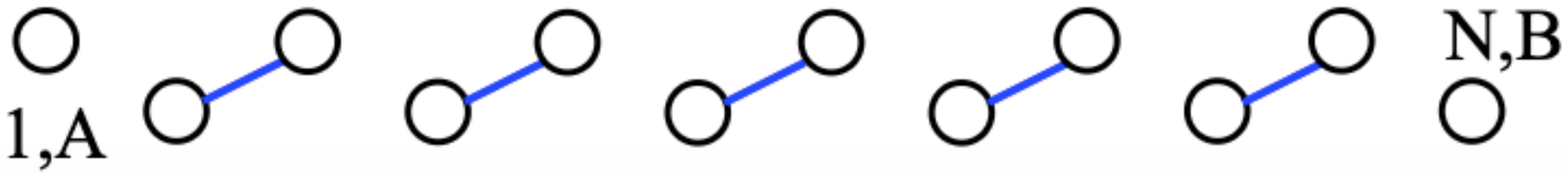
Take the topological, completely dimerized limit of the SSH model.
What is true for the state $|1,A\rangle+|N,B\rangle$?

a) Since this is a linear combination of two eigenstates, its energy is nonzero.

b) Since this is a zero energy eigenstate, it is its own chiral symmetric partner.

c) This is a zero energy eigenstate, which is not its own chiral symmetric partner.

d) Since the system is of finite length, this linear combination of states has positive energy.



Zero modes and chiral symmetry 3.

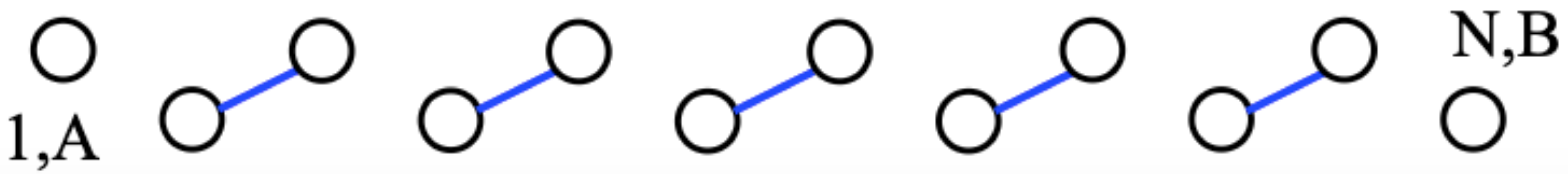
Take the topological, completely dimerized limit of the SSH model, with $v=0$ and $w=1$.
What is true for the energy eigenstates $|1,B\rangle+|2,A\rangle$ and $|2,B\rangle-|3,A\rangle$?

a) They are eigenstates of the chiral symmetry operator.

b) They are transformed into each other by the chiral symmetry operator.

c) Since they are both energy eigenstates, their arbitrary linear combination is also an energy eigenstate

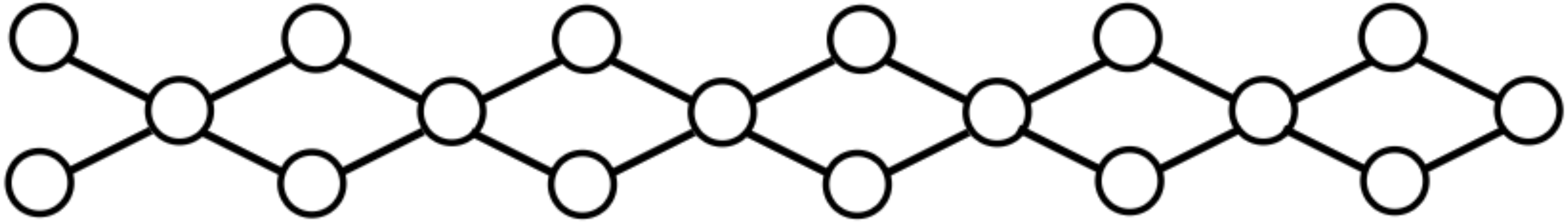
d) They are eigenstates at opposite energy, however, they are not chiral symmetric partners of each other.



Quasi-one-dimensional system 1.

We add an extra row of atoms to the SSH model
(3 rows of atoms, ABA stacking, only diagonal hoppings).
No onsite energies, only hopping.

Does this Hamiltonian have chiral symmetry?



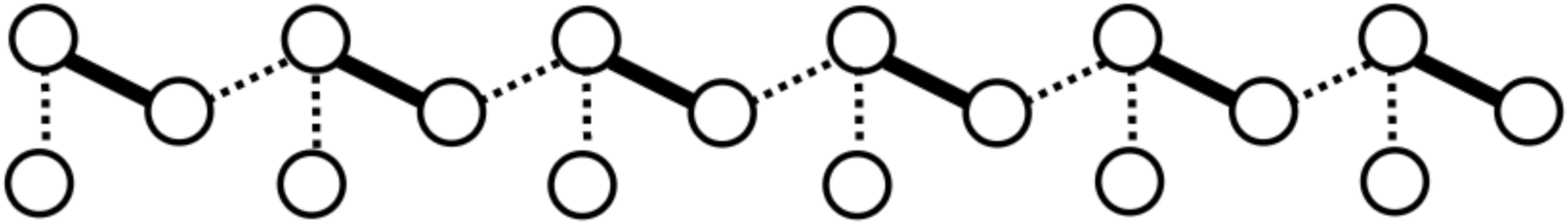
a) yes (what operator represents it?)

b) no (why?)

c) cannot decide (what does it depend on?)

Quasi-one-dimensional system 2.

We add an extra row of atoms to the SSH model. No onsite energies, only hopping.
Thick lines: hopping = 10. Thin dotted lines: hopping = 1.



Does this Hamiltonian have chiral symmetry?

a) yes (what operator represents it?)

b) no (why?)

c) cannot decide (what does it depend on?)

Complex hopping 2.

In the SSH model with real valued nearest neighbor hopping there are two topological classes. How many topological classes do we have if we allow for a real valued third neighbor hopping, as depicted by green and magenta lines?

a) 4

b) 1

c) 2

d) 3