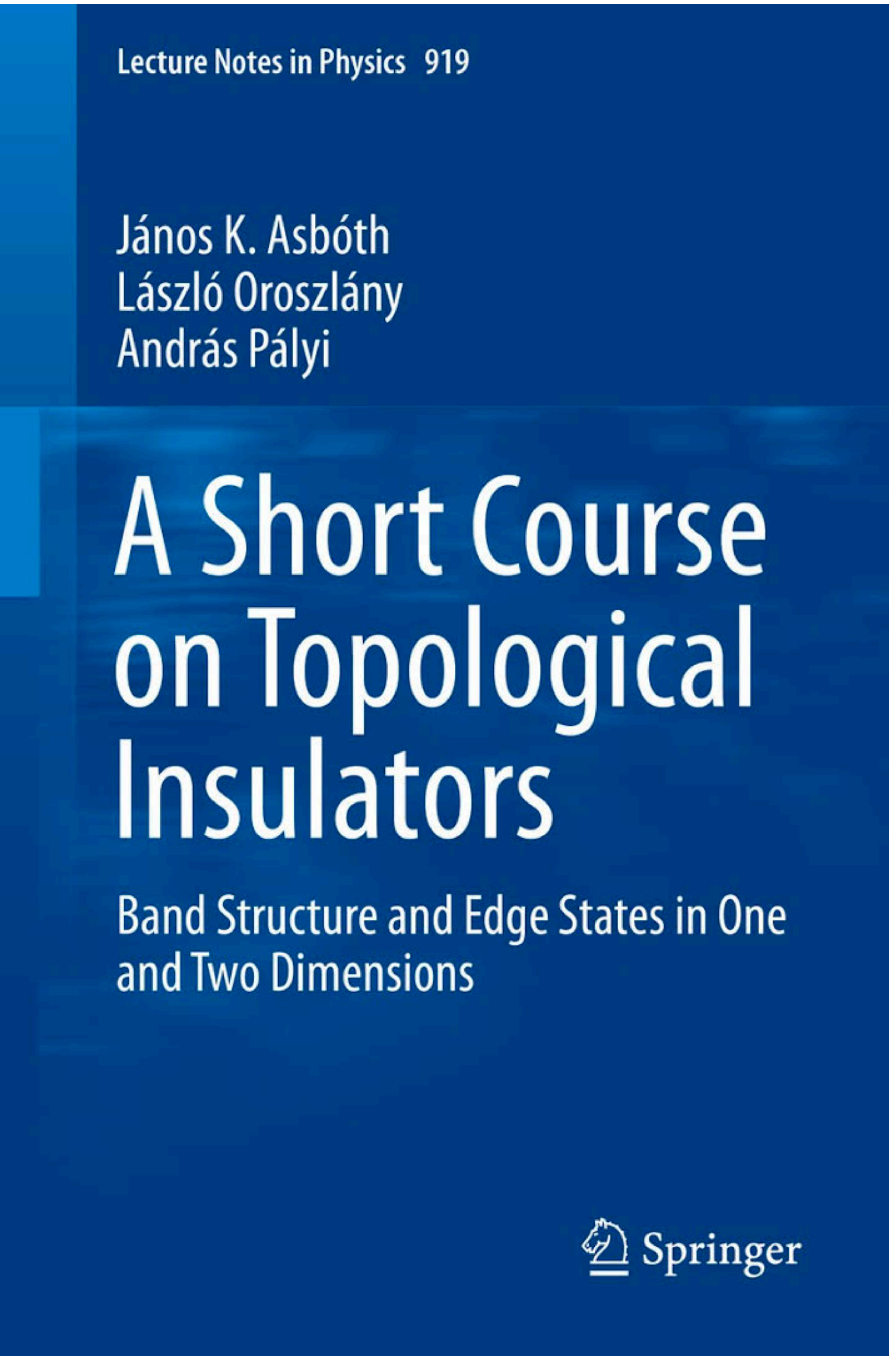
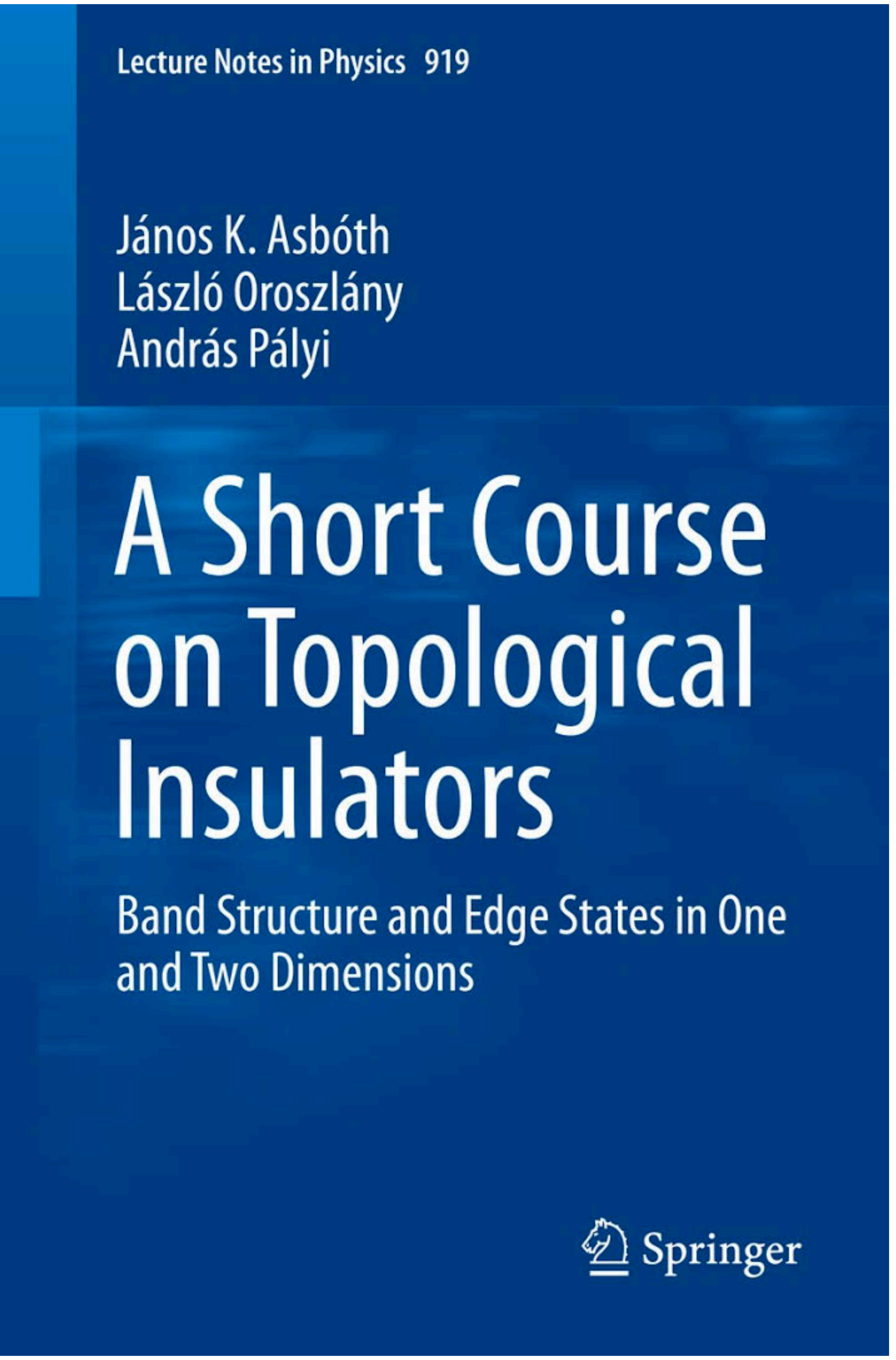
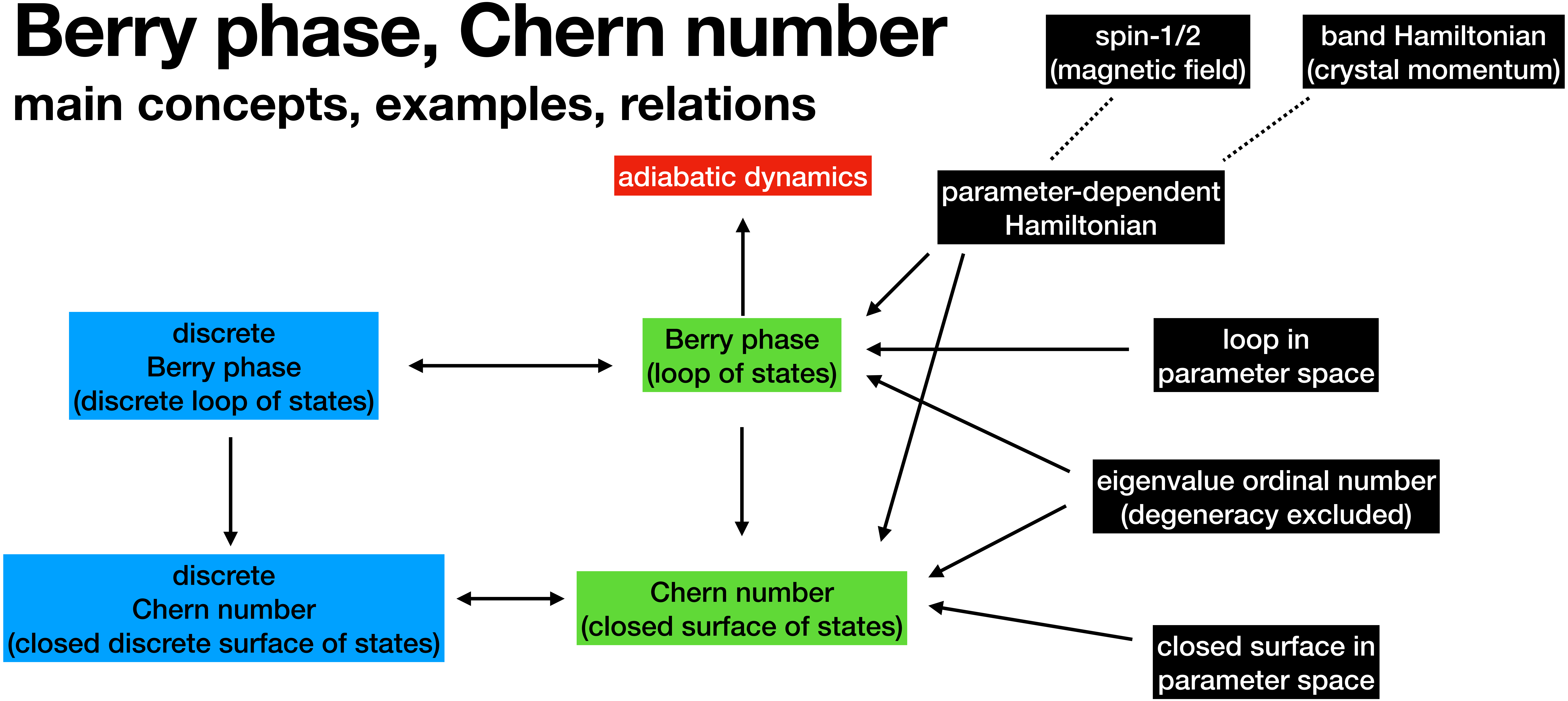

Lecture Notes in Physics 919
János K. Asbóth
László Oroszlány
András Pályi
A Short Course on Topological Insulators
Band Structure and Edge States in One and Two Dimensions
Springer

For next week, please read Chapters 3 and 4



Berry phase, Chern number

main concepts, examples, relations

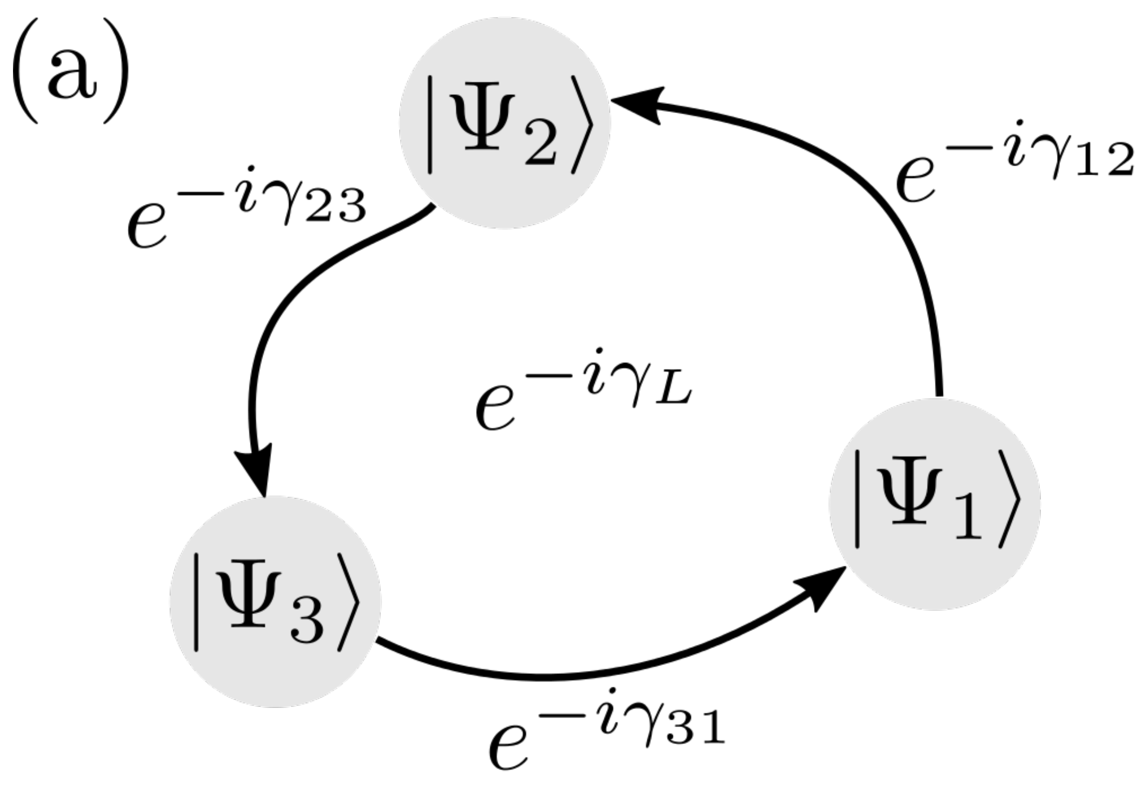


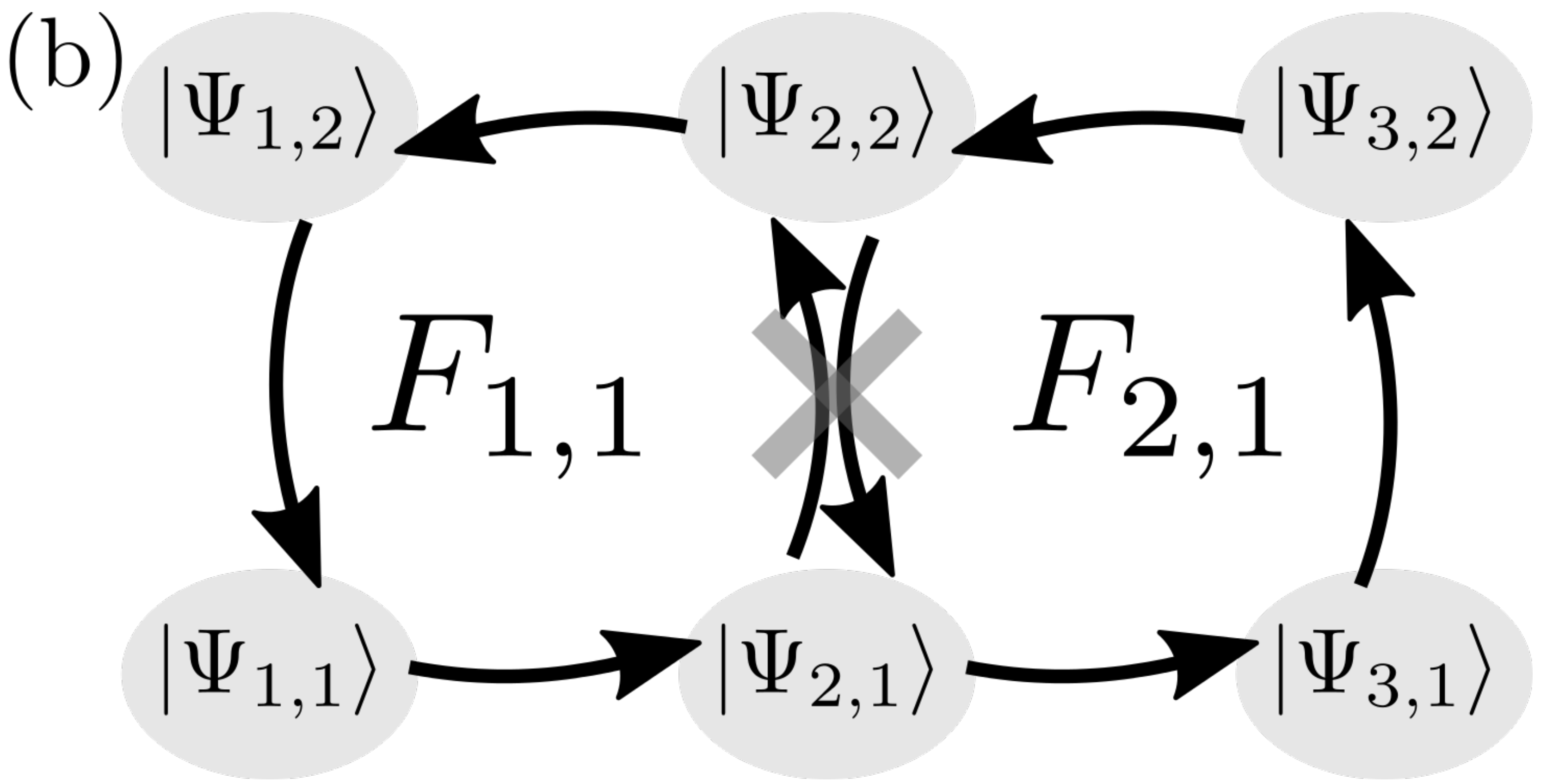
remark: terminology: *Berry phase factor* lives on the complex unit circle, *Berry phase* lives in]-pi,pi] or [0,2pi[

Berry phase, Chern number

main concepts, examples, relations

discrete
Berry phase
(discrete loop of states)





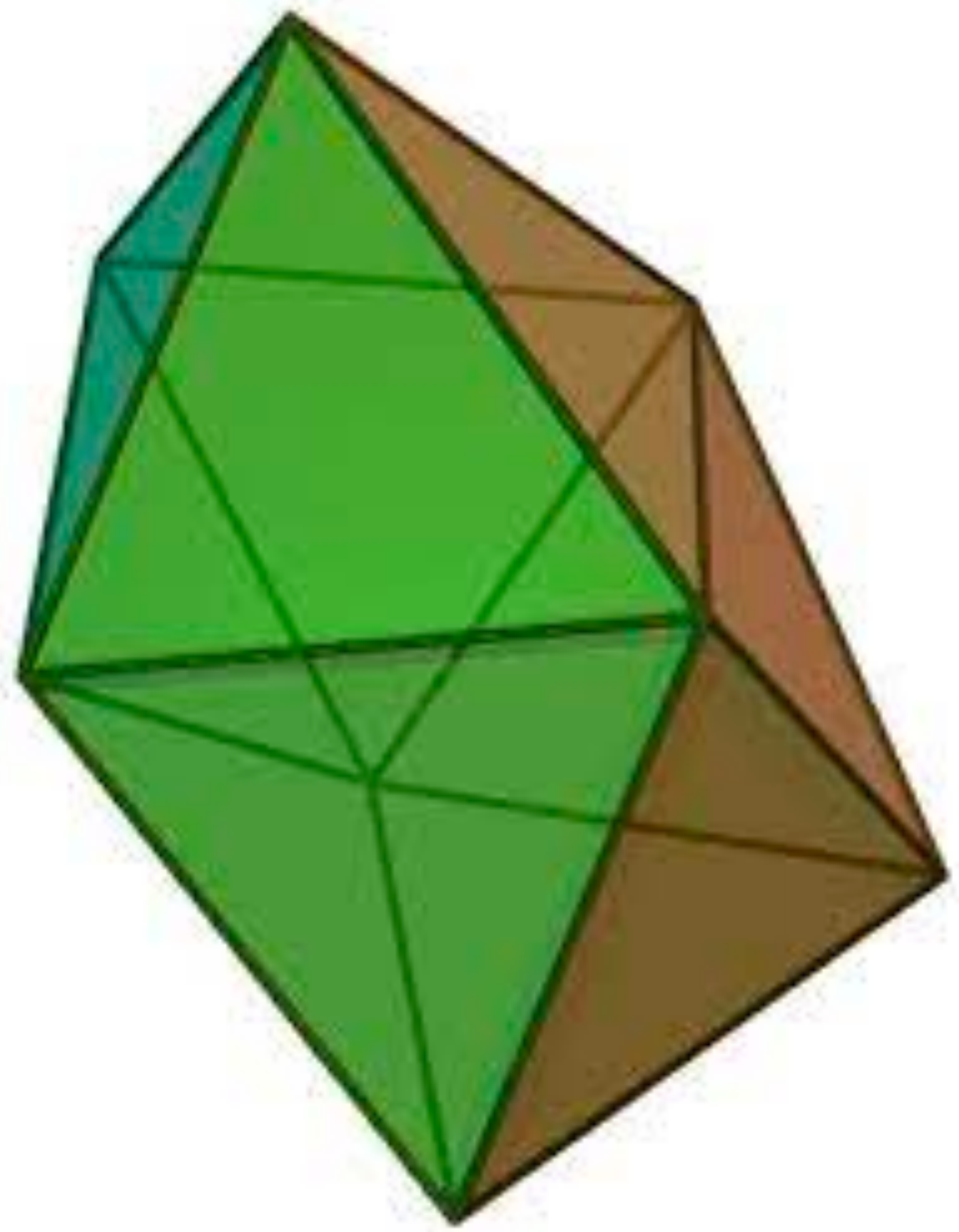
Berry phase, Chern number

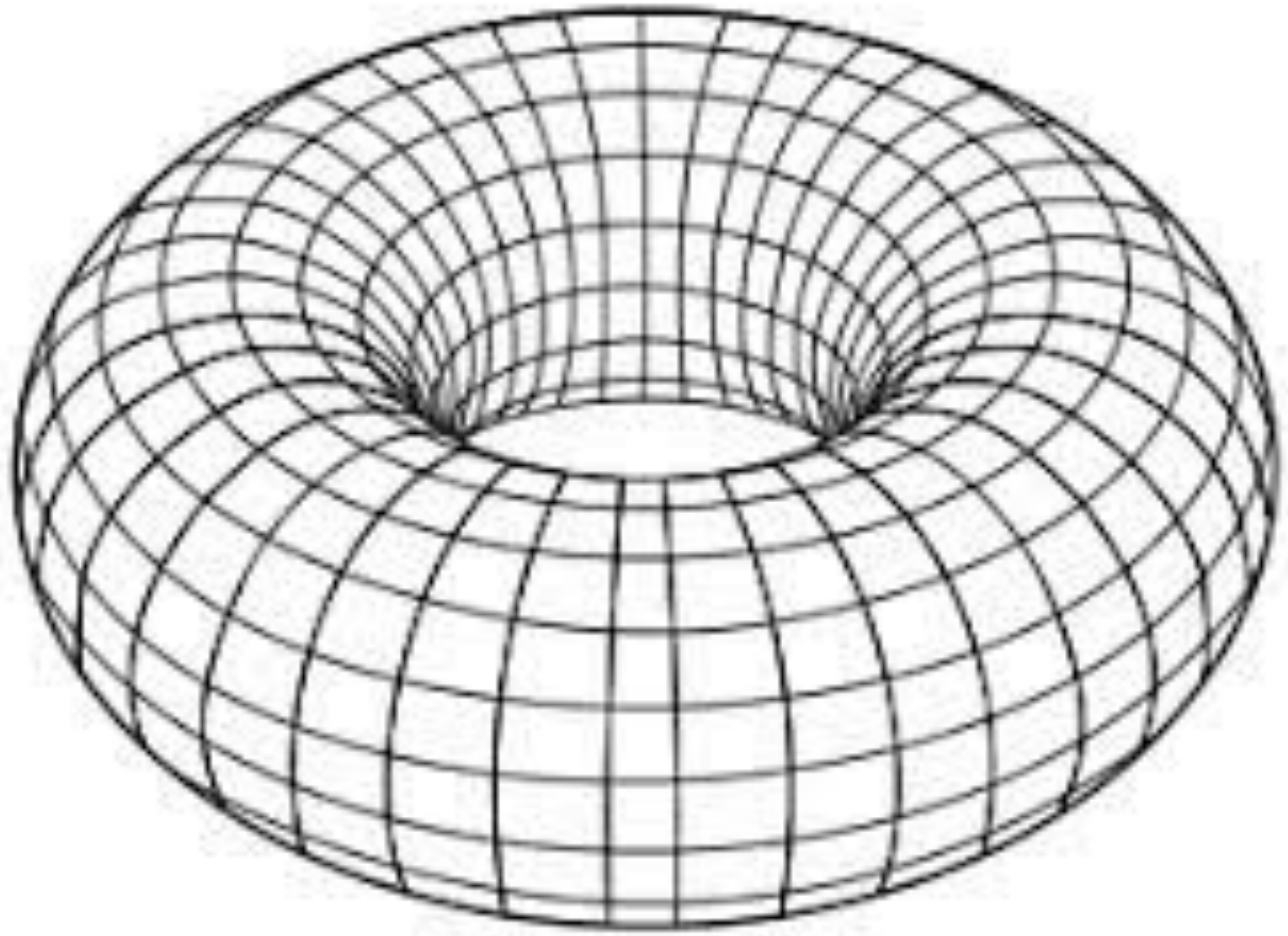
main concepts, examples, relations

discrete
Berry phase
(discrete loop of states)

↓

discrete
Chern number
(closed discrete surface of states)





Berry phase, Chern number

main concepts, examples, relations

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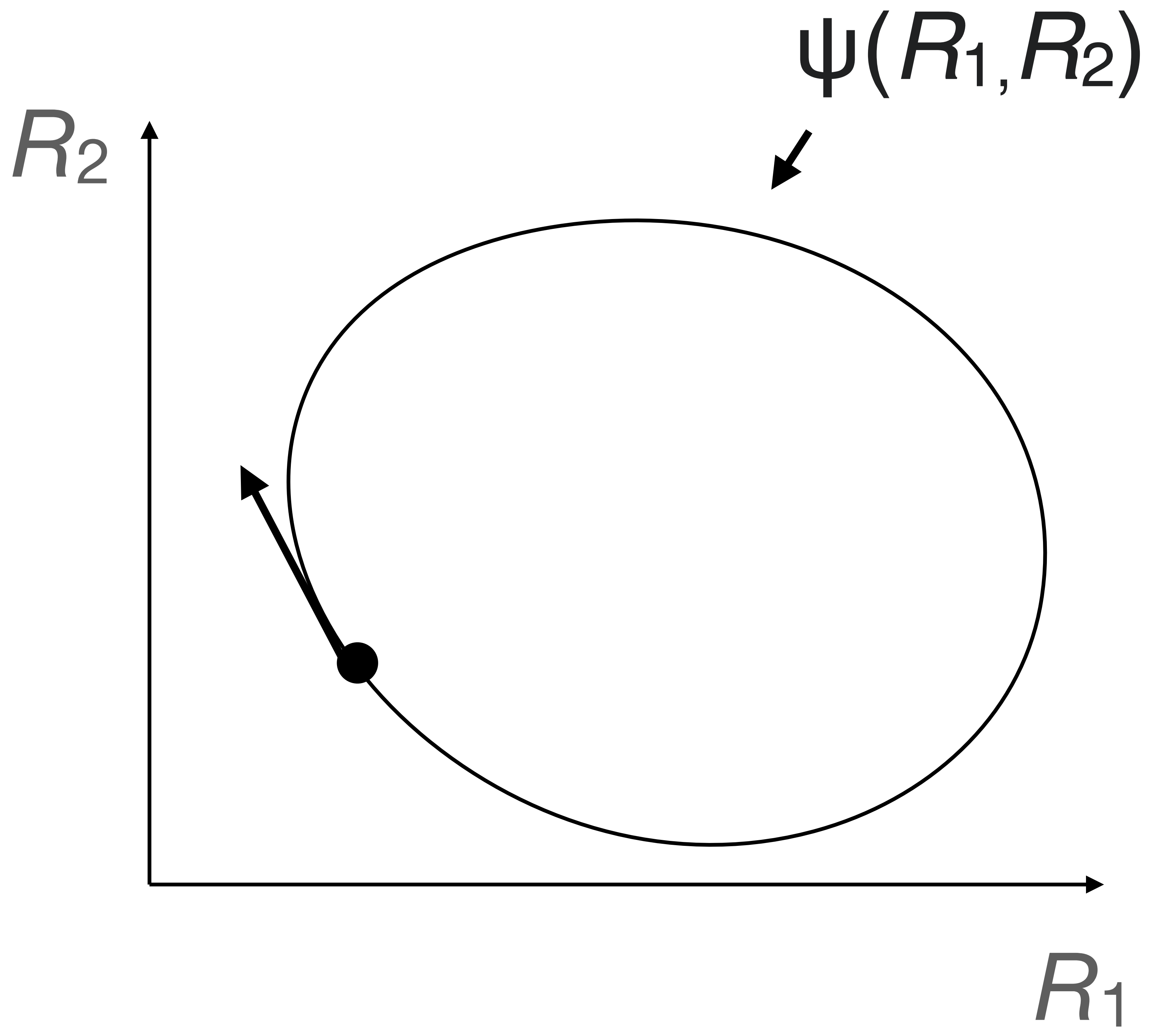

Berry phase, Chern number

main concepts, examples, relations

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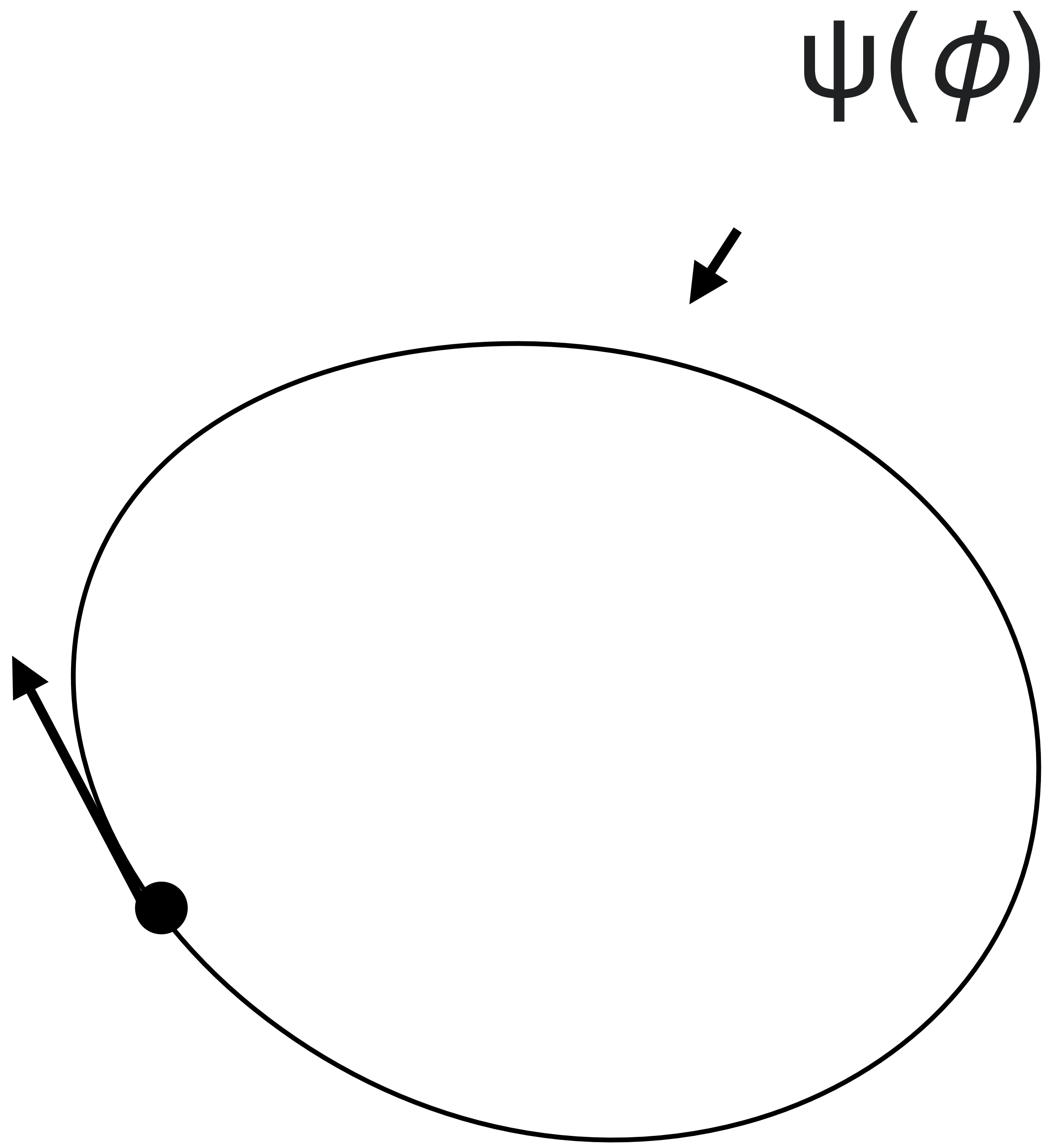
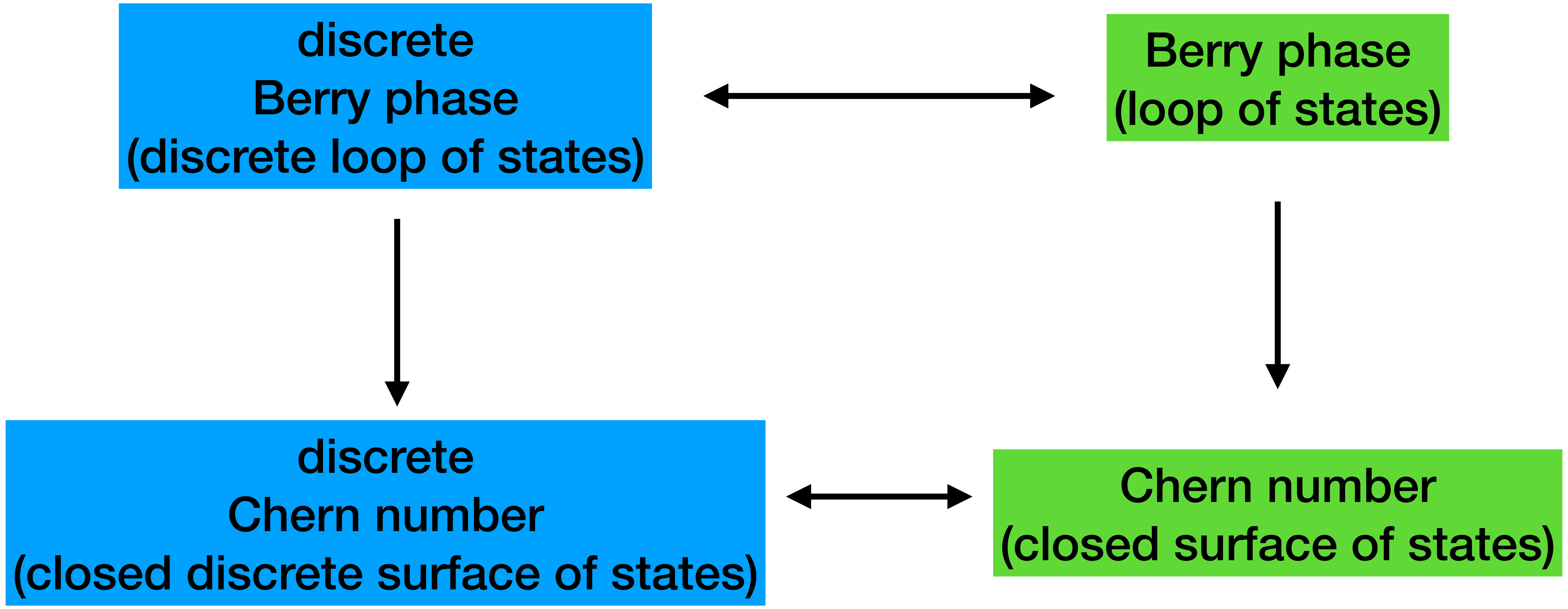
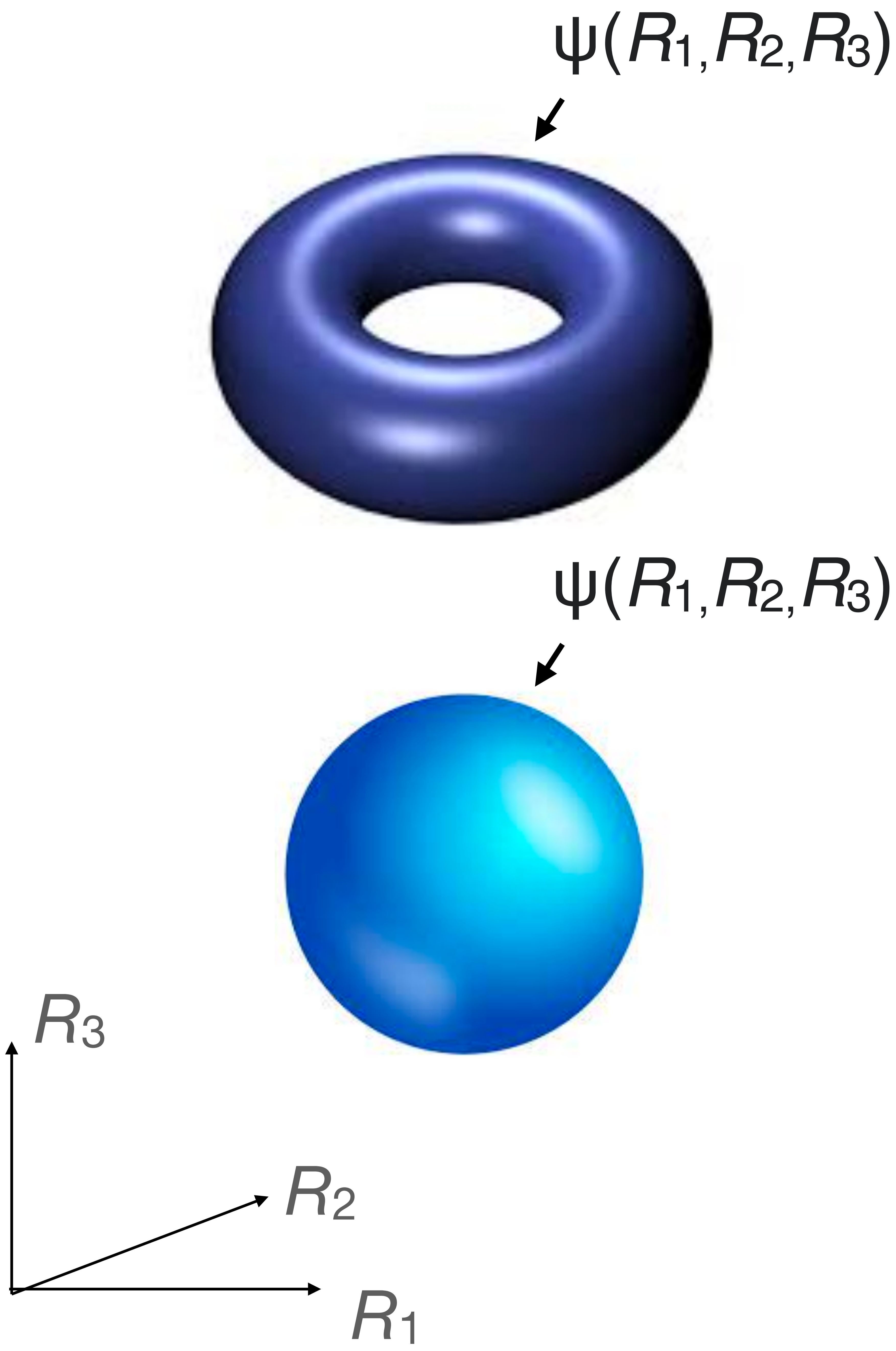


Berry phase, Chern number

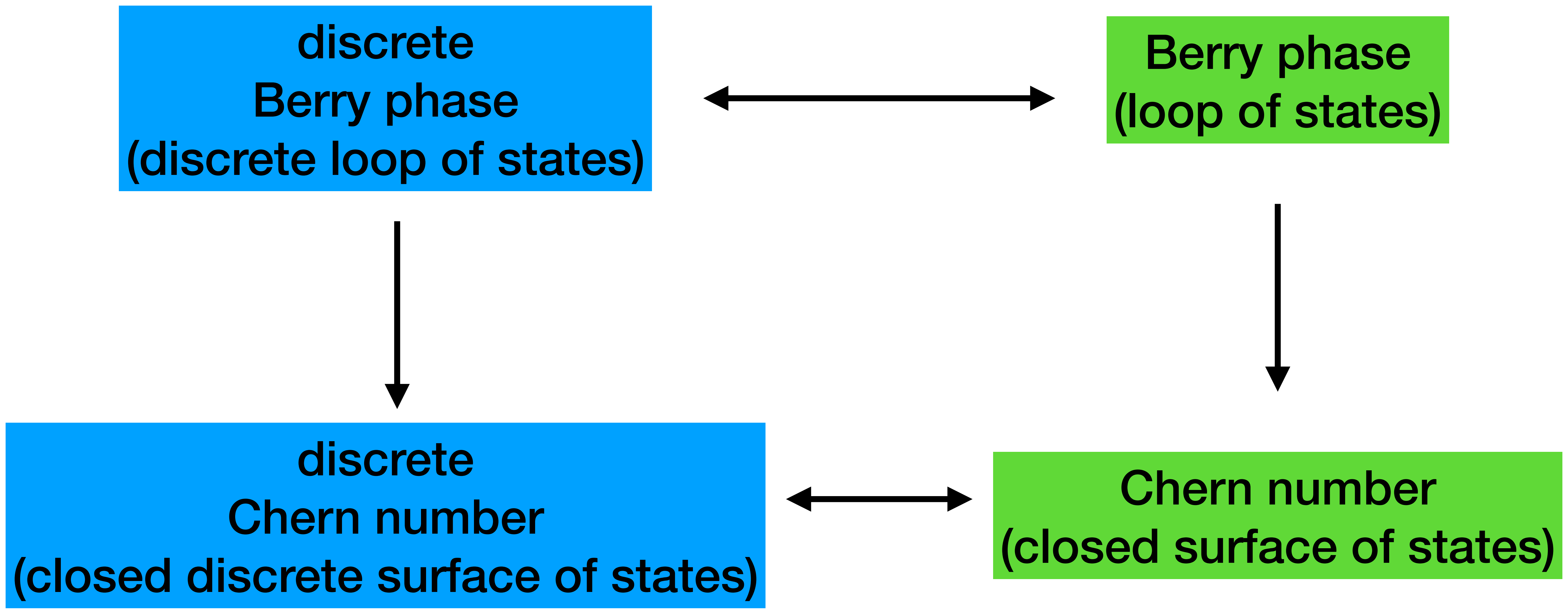
main concepts, examples, relations

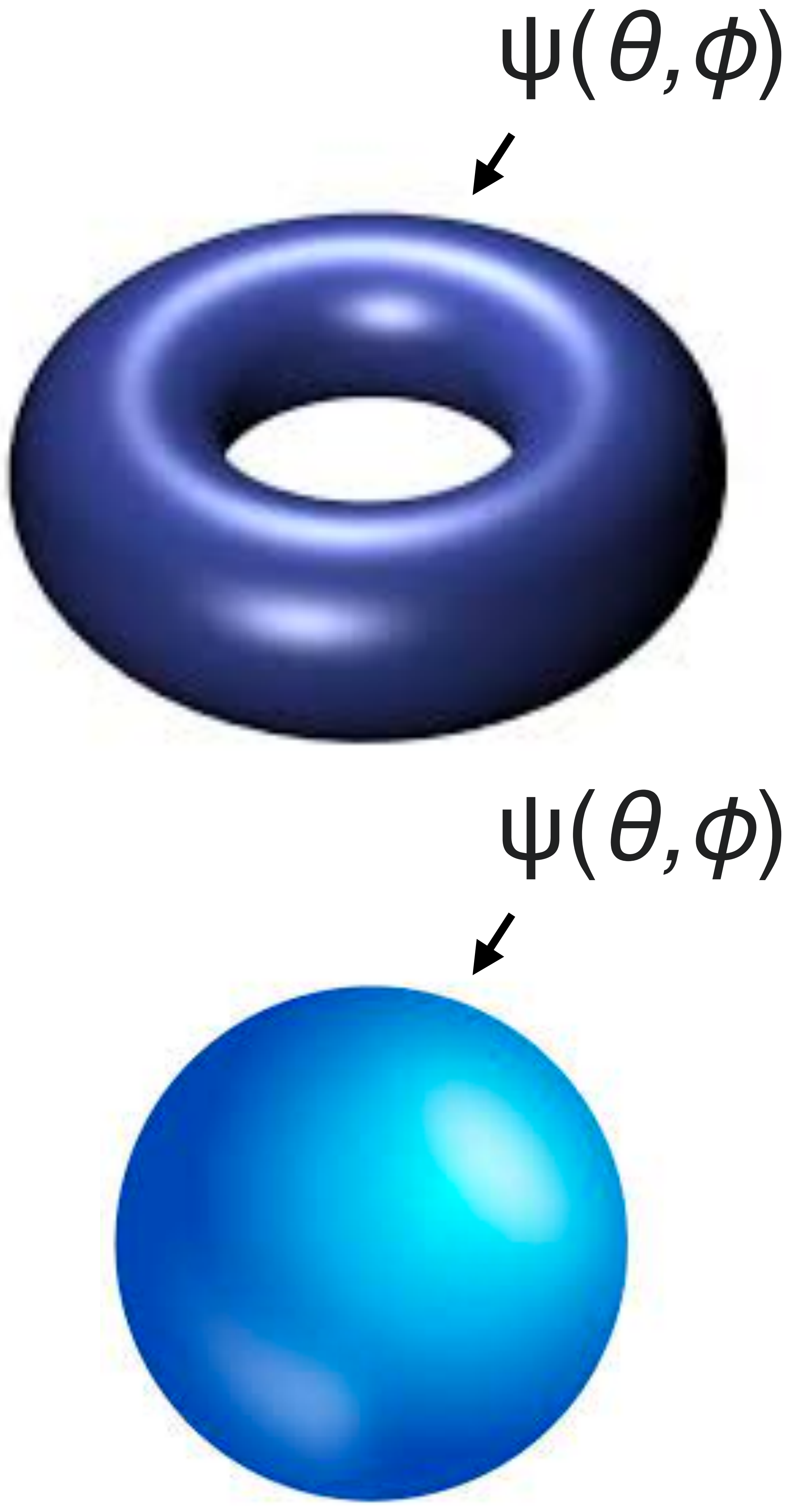




Berry phase, Chern number

main concepts, examples, relations





Berry phase, Chern number

main concepts, examples, relations

spin-1/2
(magnetic field)

band Hamiltonian
(crystal momentum)

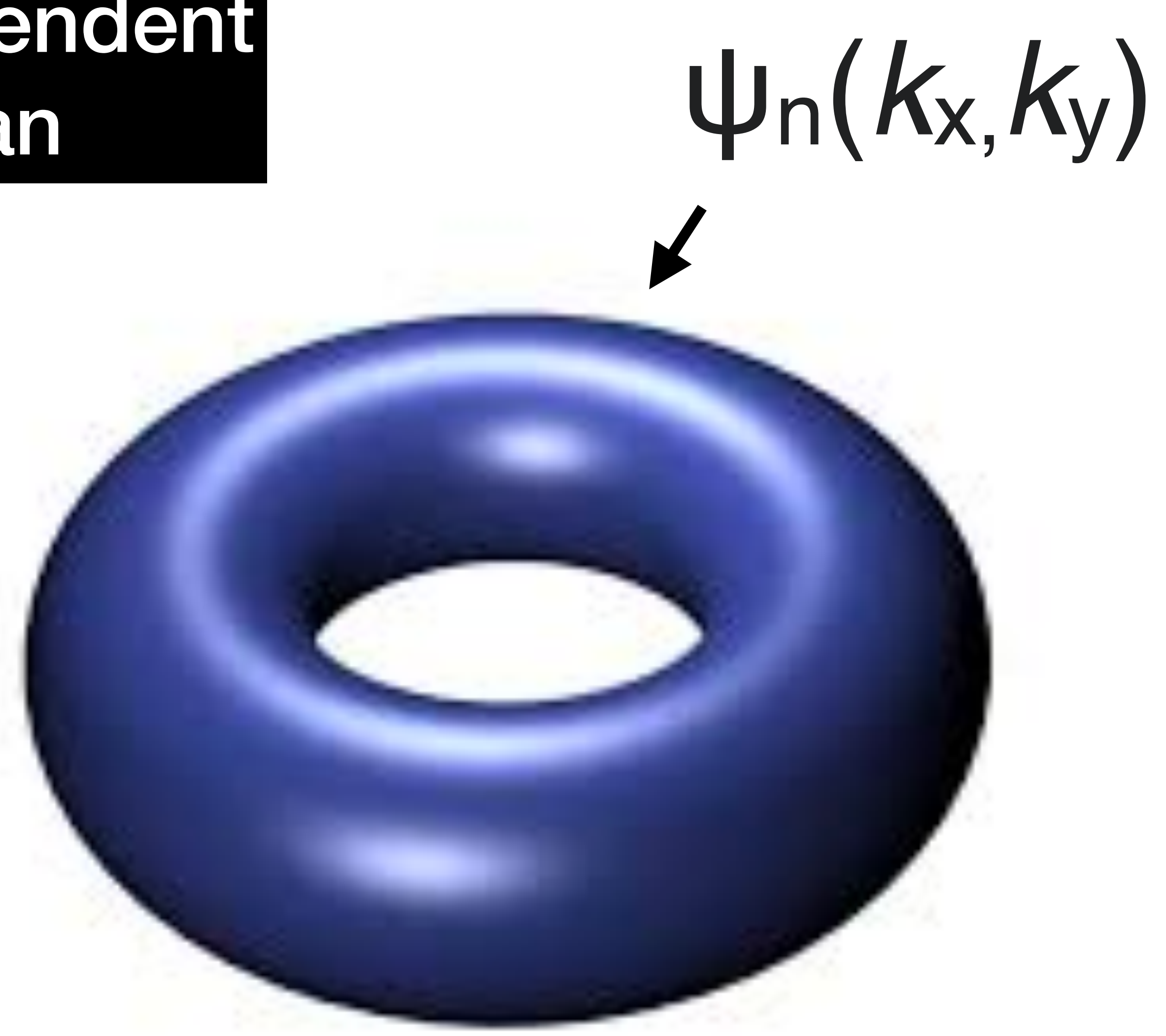
parameter-dependent
Hamiltonian

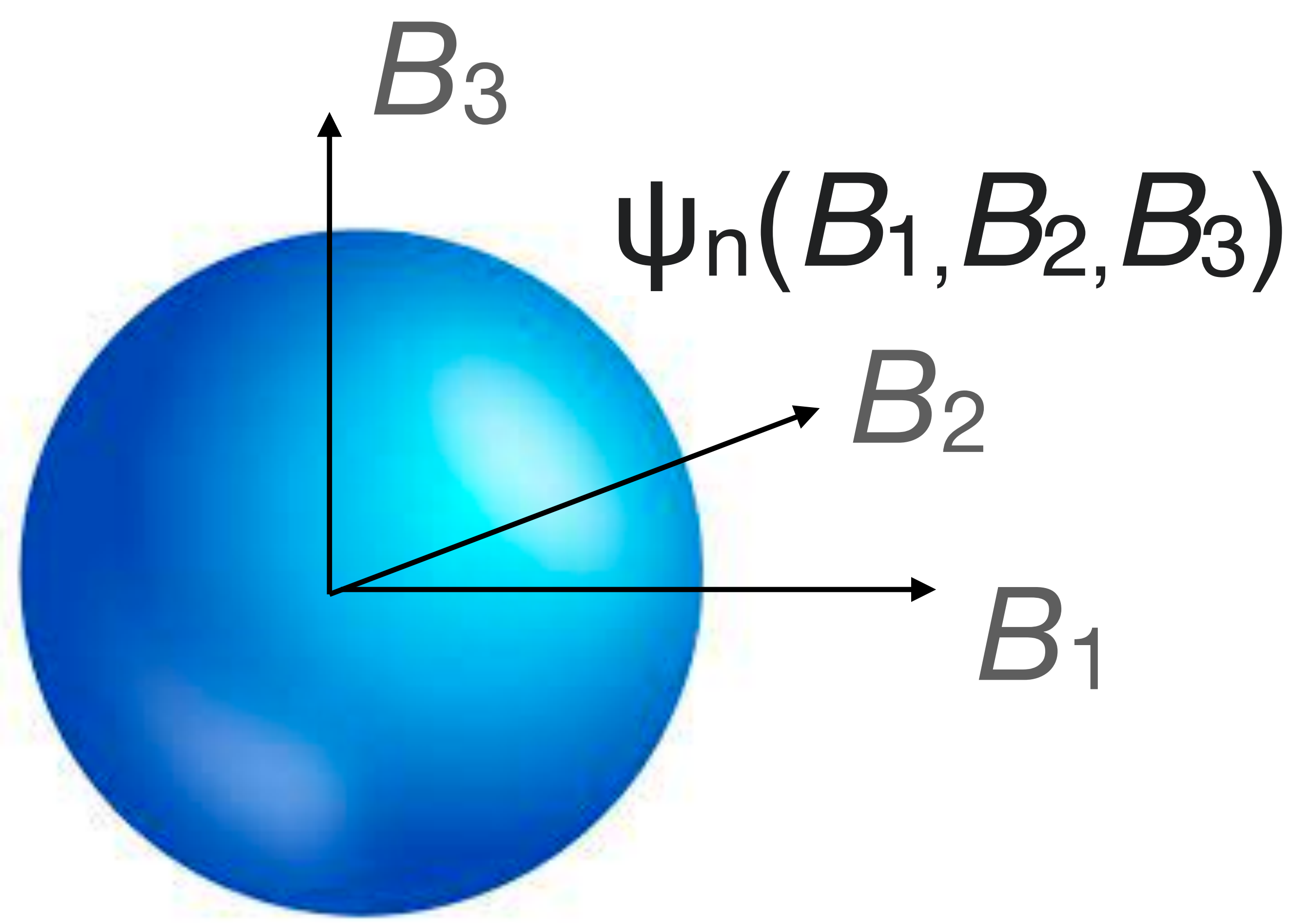
discrete
Berry phase
(discrete loop of states)

Berry phase
(loop of states)

discrete
Chern number
(closed discrete surface of states)

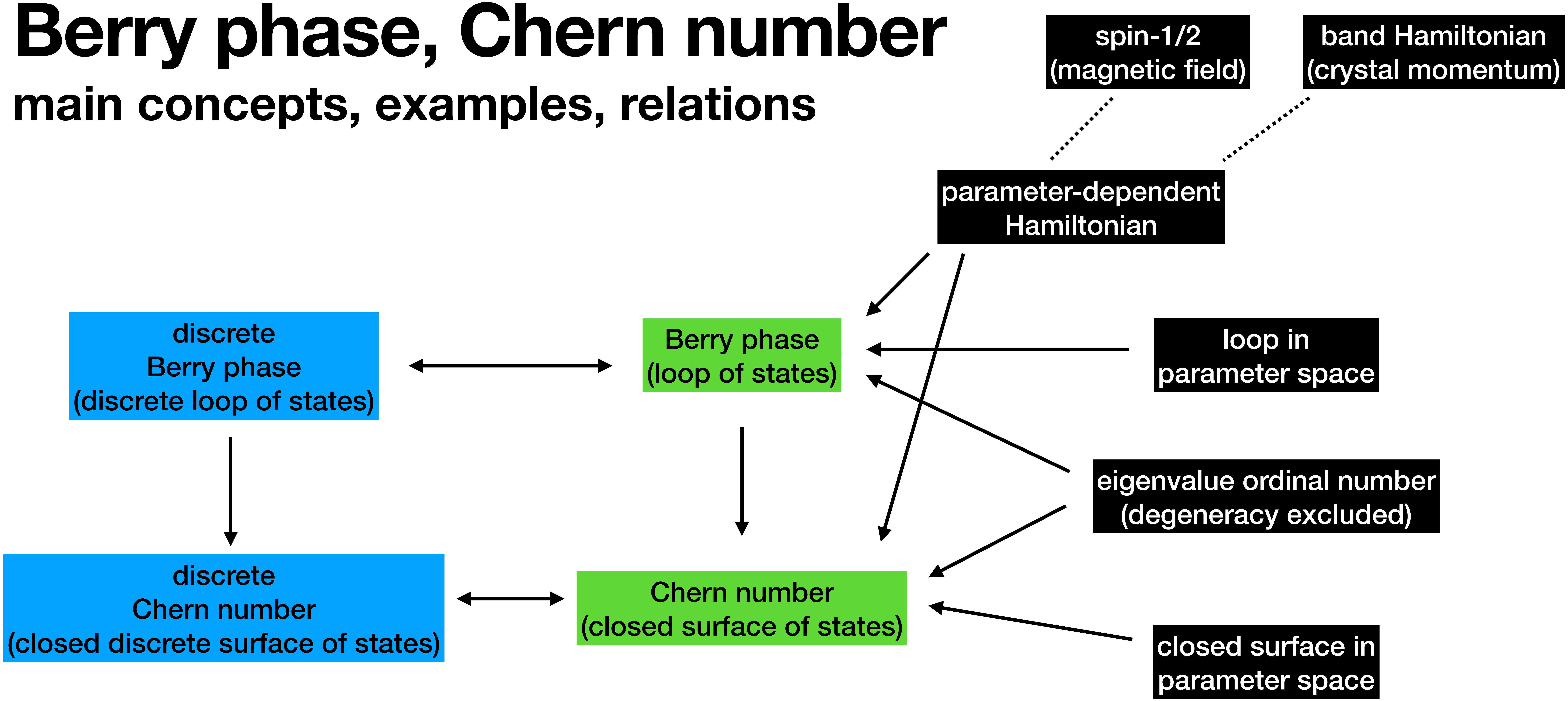
Chern number
(closed surface of states)





Berry phase, Chern number

main concepts, examples, relations



Berry phase, Chern number

main concepts, examples, relations

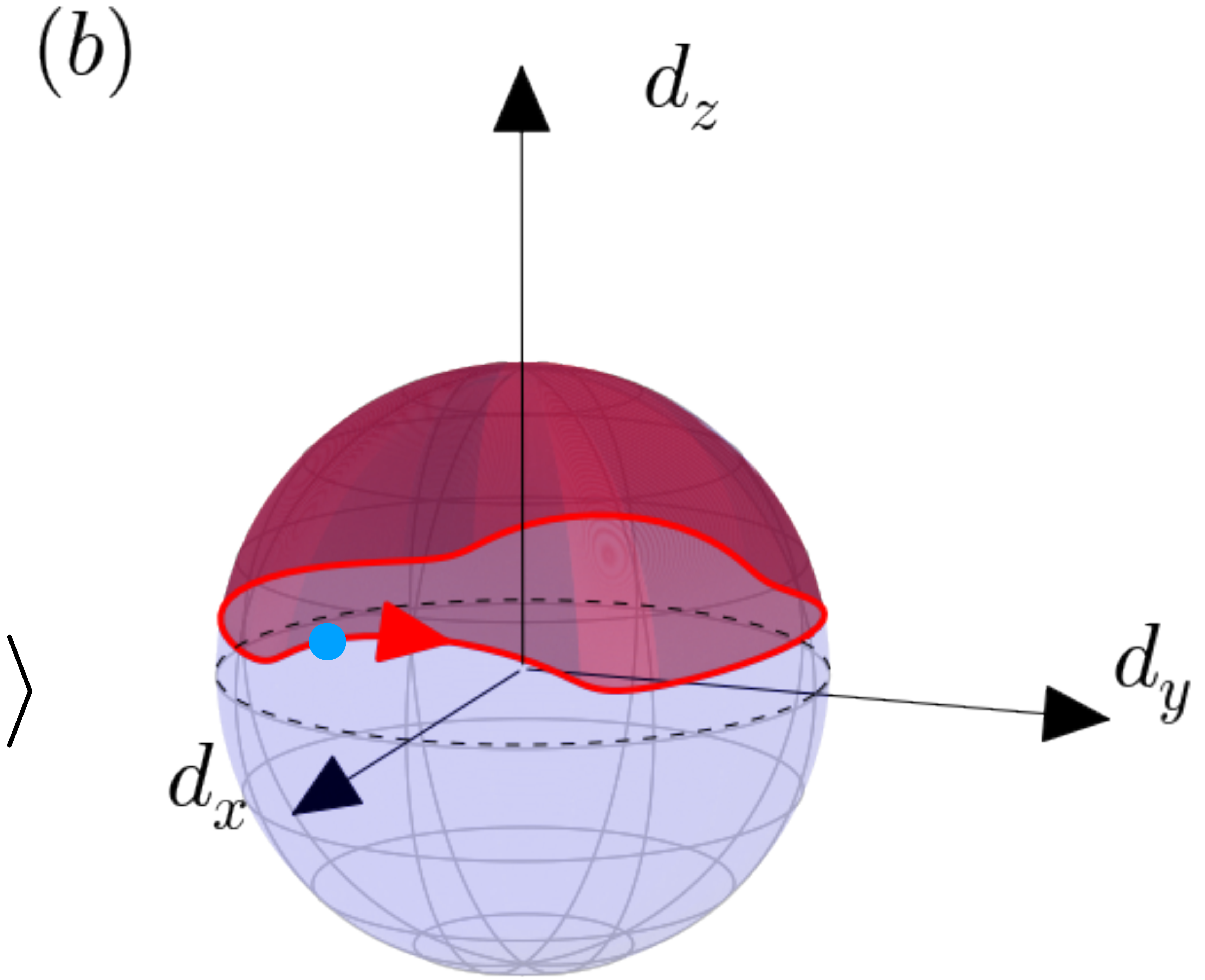
adiabatic dynamics

↑

Berry phase
(loop of states)

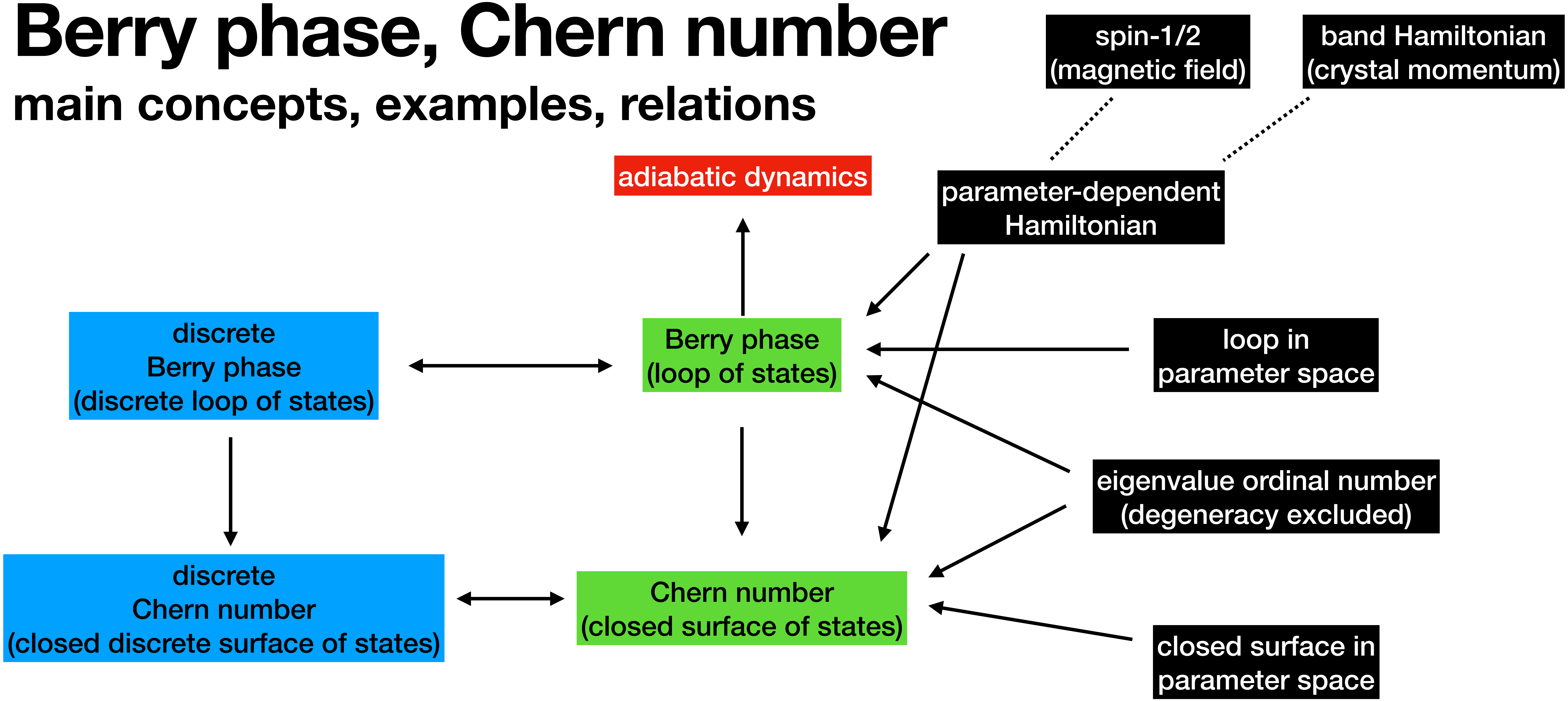
$$|\psi(t)\rangle = e^{i\gamma_n(t)} e^{-i\int_0^t E_n(\mathbf{R}(t'))dt'} |n(\mathbf{R}(t))\rangle$$

$$\gamma_n(\mathscr{C}) = \oint_{\mathscr{C}} i \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} n(\mathbf{R})\rangle \, d\mathbf{R}$$



Berry phase, Chern number

main concepts, examples, relations



remark: terminology: *Berry phase factor* lives on the complex unit circle, *Berry phase* lives in]-pi,pi] or [0,2pi[

Berry phase definition 1

Take a 3-parameter Hamiltonian $H(R_1,R_2,R_3)$,
with a nondegenerate spectrum

The Berry phase of the n-th energy eigenstate is a mapping...

a) from the set of closed curves in the parameter space
to the interval $[0,2\pi[$

b) from the set of points of the parameter space
to the set of real numbers

c) from the set of open curves in the parameter space
to the interval $[0,2\pi[$

d) from the set of open curves in the parameter space
to the set of real numbers

Berry phase definition 2

Take a quantum system parametrized by the continuous $\mathbf{R}$, which obeys the Schrodinger equation $H(\mathbf{R})|n(\mathbf{R})\rangle = e_n(\mathbf{R})|n(\mathbf{R})\rangle$
What is characterized by an adiabatic phase?

a) The operator $H(\mathbf{R})$ along a closed curve in the parameter space

b) The energy eigenvalues $e_n(\mathbf{R})$ at a given point $\mathbf{R}_0$

c) A state vector $|n(\mathbf{R})\rangle$ along a continuous curve in the parameter space

d) the linear combination of state vectors $|n(\mathbf{R})\rangle, |m(\mathbf{R})\rangle$ at two points $\mathbf{R}_1$ és $\mathbf{R}_2$ connected by a continuous curve

Berry phase, gauge invariance

Which of these is gauge invariant?

a) adiabatic phase $$\gamma_n(\mathcal{C}) = \mathrm{i} \int_{\mathcal{C}} \langle n(\mathbf{R}) | \nabla_{\mathbf{R}}\, n(\mathbf{R}) \rangle \, \mathrm{d}\mathbf{R}$$

b) Berry connection $$\mathbf{A}^n = \mathrm{i} \, \langle n(\mathbf{R}) | \nabla_{\mathbf{R}}\, n(\mathbf{R}) \rangle$$

c) Berry curvature $$\mathbf{B}^n = \nabla_{\mathbf{R}} \times \mathbf{A}^n$$

Berry phase in SSH

Take the SSH model with hopping amplitudes *v* and *w* both real.
When will the Berry phase of the lower band of the system be 0?

$$\hat{H}_{\mathrm{SSH}}(k) = v\hat{\sigma}_x + w\cos(k)\hat{\sigma}_x + w\sin(k)\hat{\sigma}_y$$

a) *v*>*w*

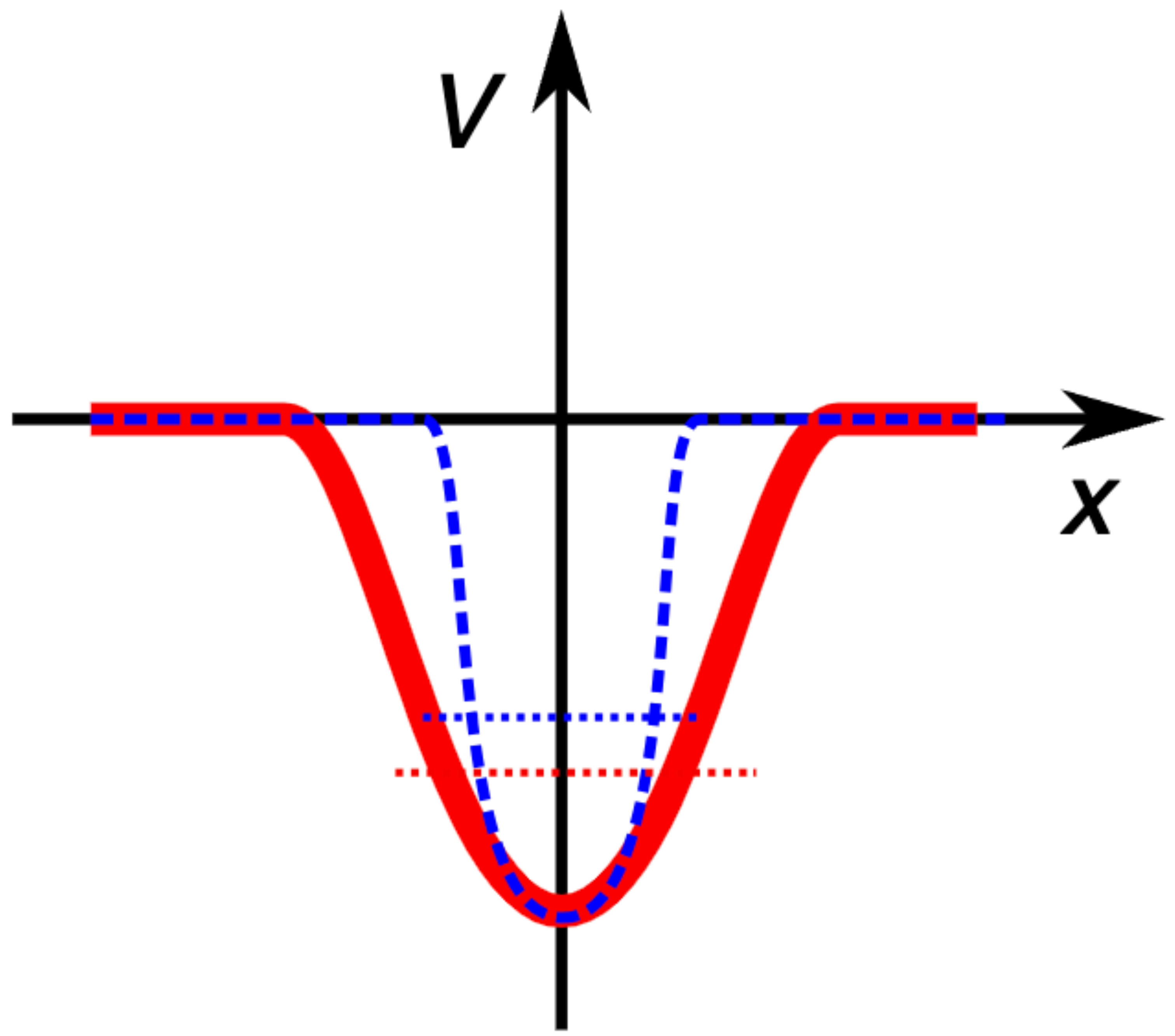
b) *v*=*w*

c) *v*<*w*

d) not enough information to decide

Berry cyclic 1.

We adiabatically slowly compress a 1D potential well (continuous thick red) until its width is reduced by a factor of 1/2 (thin blue dashed).
We then decompress it back.
$V(t+T,x)=V(t,x)$
$V(T/2-t,x)=V(T/2+t,x)$
What is the Berry phase accumulated by the lowest energy bound state ?



a) There is no Berry phase defined to this process

b) 2π

c) 0

d) The answer depends on the precise shape of the potential.

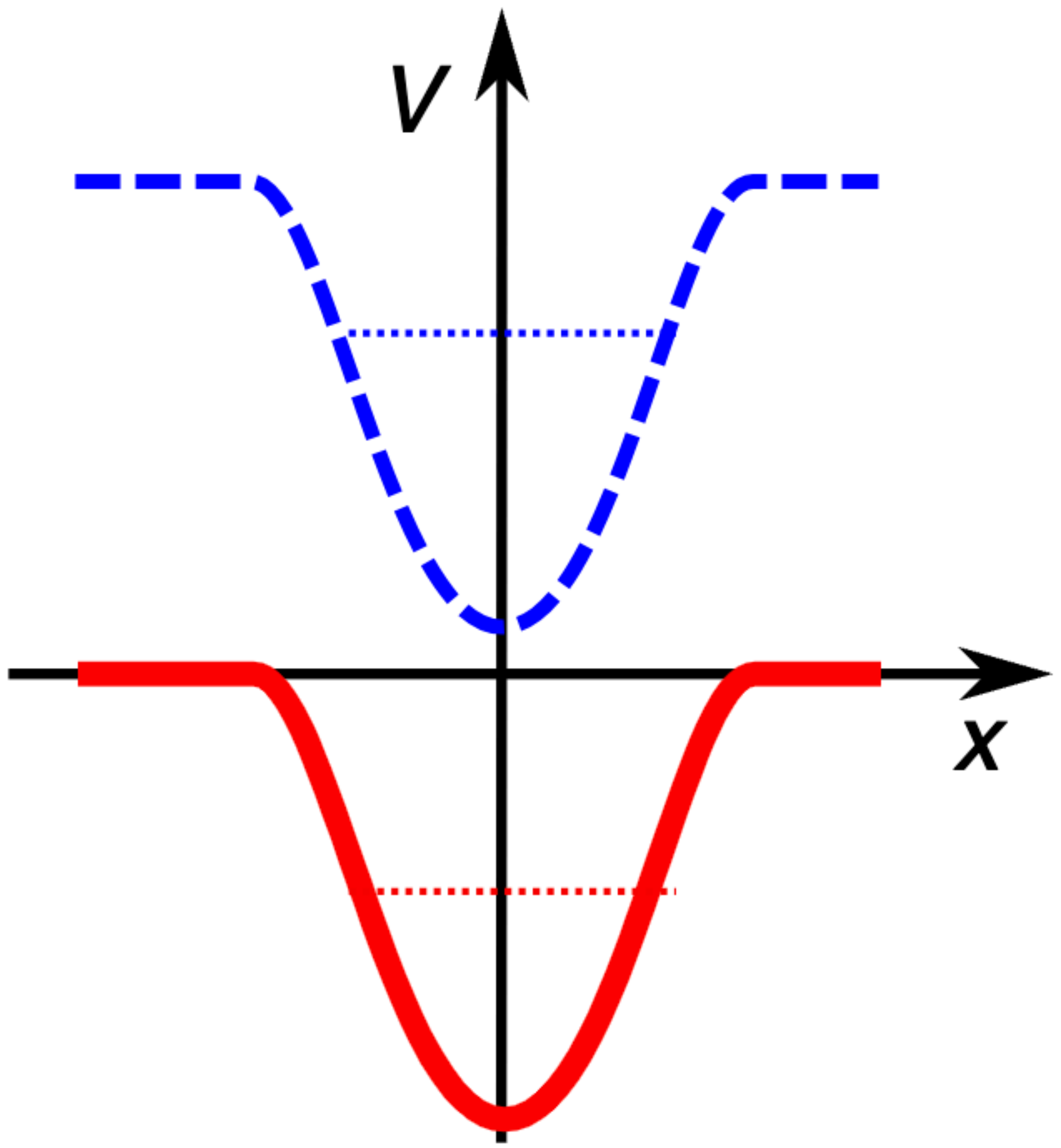
Berry cyclic 2.

We adiabatically slowly lift a 1D potential well (continuous thick red) until at every x it is positive (thin blue dashed). We then lower it back.

$V(t+T,x)=V(t,x)$

$V(T/2-t,x)=V(T/2+t,x)$

What is the Berry phase accumulated by the lowest energy bound state ?



a) Since raising the potential will increase the energy of the bound state well above its original confinement, the particle will not remain a bound state.

b) 2π

c) 0

d) Can not be decided.

Bloch sphere 1.

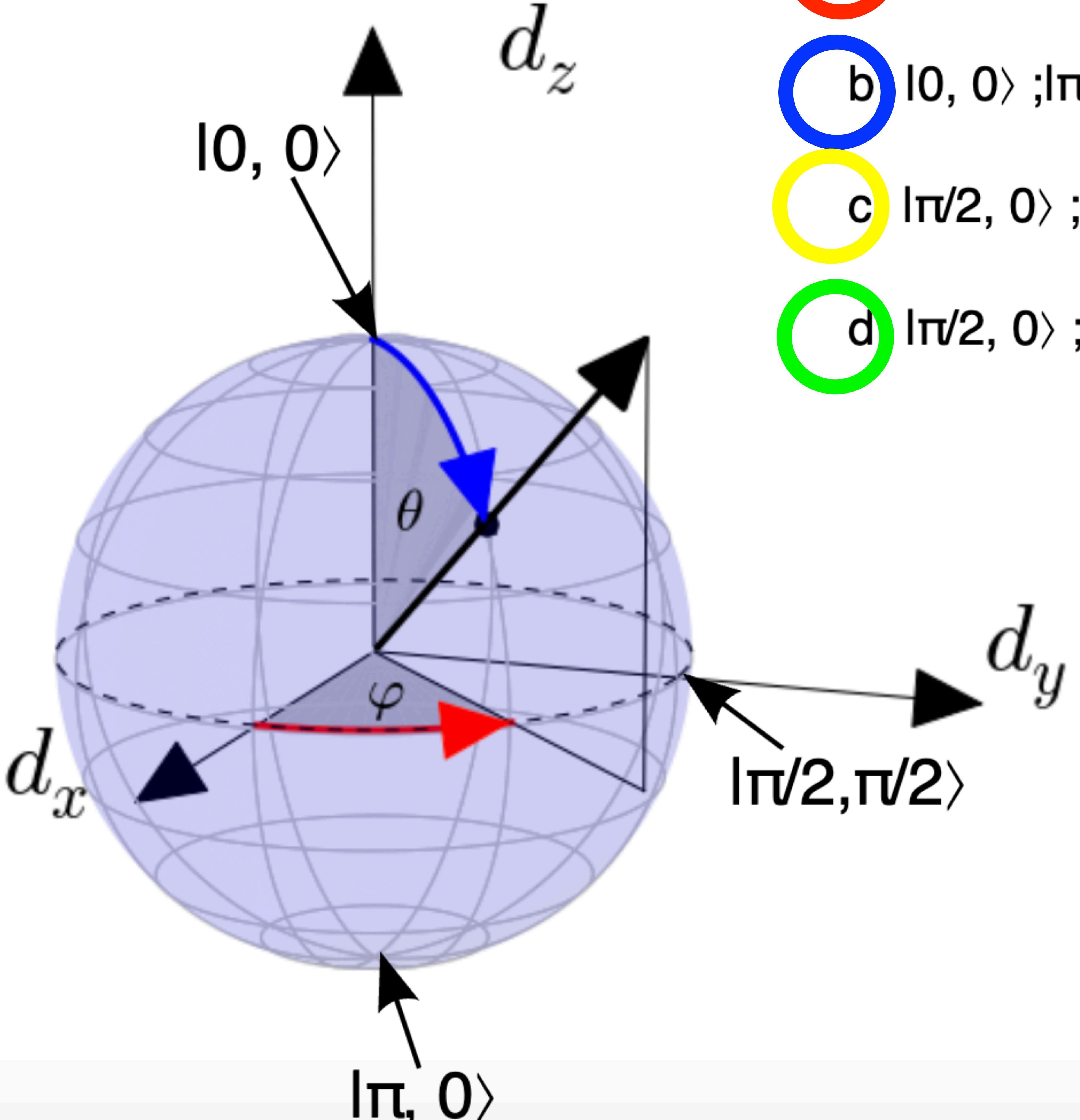
Let us denote states on the Bloch sphere by |θ, ϕ〉 ! Which sequence of states has a finite Berry phase associated to it?

a) |0, 0〉 ;|0, π/2〉 ;|0, π〉; |0, 3π/2〉;|0,2 π〉

b) |0, 0〉 ;|π/4, 0〉 ;|π/2, 0〉; |π/4, 0〉;|0,0〉

c) |π/2, 0〉 ;|π/2, 2π/3〉 ;|0, π〉; |π/4, -π/7〉;|π/2,0〉

d) |π/2, 0〉 ;|π/2, 2π/3〉 ;|0, π〉; |π/4, -π/7〉;|π,0〉



Take the Hamiltonian H(**R**) defined in a three-dimensional parameter space, and consider its ground-state manifold $\psi_0(\mathbf{R})$.
Assume that the ground-state manifold is non-degenerate, $E_0(\mathbf{R}) < E_j(\mathbf{R})$ for any $j>0$.
Then the Chern number of the ground-state manifold is a function that maps

a) closed loops in the parameter space to integer numbers

b) closed loops in the parameter space to $[0,2\pi[$

c) closed surfaces in the parameter space to integer numbers

d) closed surfaces in the parameter space to real numbers

Take the Hamiltonian $H(\mathbf{B}) = - B_z \sigma_z$.
What is the Chern number associated to the ground-state manifold on the sphere $|\mathbf{B}| = B_0$?

a) 0

b) 1

c) 2

d) undefined

Take the Hamiltonian H(**R**) defined in a three-dimensional parameter space. Assume that the ground-state manifold is non-degenerate on the sphere $\mathbf{R} = \mathbf{R}_0$, and that the Chern number associated to this ground-state manifold on this sphere is 1.
Which is the most precise statement about the ground-state degeneracies in the interior of this sphere?

a) There is no ground-state degeneracy.

b) There is exactly one degeneracy point, and there the ground state is twofold degenerate.

c) There is exactly one point where the ground state is degenerate.

d) There is at least one point where the ground state is degenerate.

Take the Hamiltonian $H(\mathbf{R})$ defined in a three-dimensional parameter space. Assume that the ground-state manifold is non-degenerate on the sphere $|\mathbf{R}| = R_0$, and that the Chern number associated to this ground-state manifold on this sphere is 0.
Which is the most precise statement about the ground-state degeneracies in the interior of this sphere?

a) There is no ground-state degeneracy.

b) There is exactly one point where the ground state is degenerate.

c) The number of degeneracy points is even.

d) None of the above is true in general.