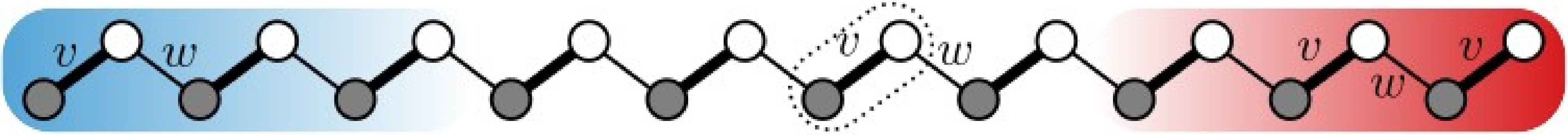
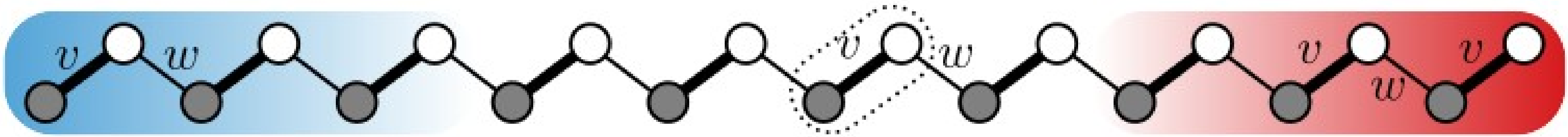


Topological insulators

1. SSH
Bulk, edge, topology, bulk-edge correspondence

Budapest, 2023 September 18

SSH model: simplest 2-band insulator. External vs internal degrees of freedom



$$\hat{H} = v\sum_{m=1}^{N} |m,A\rangle\langle m,B| + h.c. + w\sum_{m=1}^{N-1} |m+1,B\rangle\langle m,A| + h.c.$$

1 unit cell = two sites, intracell/intercell hopping v/w
External degree of freedom: which unit cell, m=
Internal degree of freedom: which site in cell, sublattice

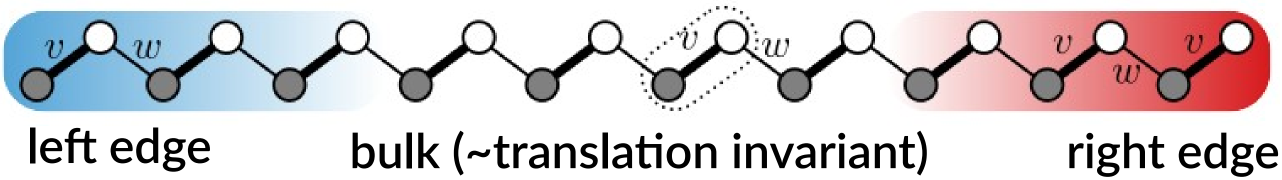
$$|m,\alpha\rangle \to |m\rangle \otimes |\alpha\rangle \in \mathscr{H}_{\text{external}} \otimes \mathscr{H}_{\text{internal}},$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

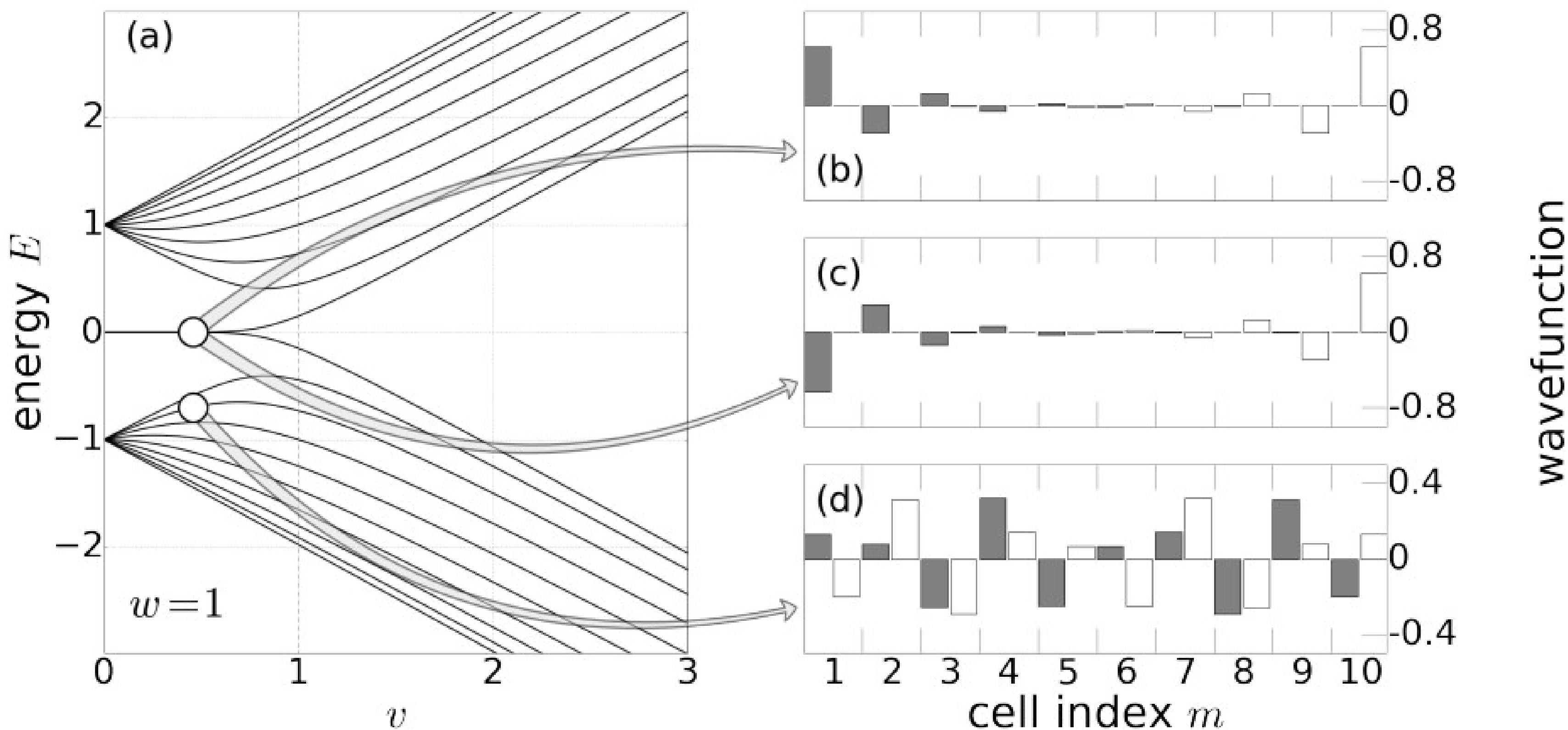
Rewrite Hamiltonian:

$$\hat{H} = v\sum_{m=1}^{N} |m\rangle\langle m| \otimes \hat{\sigma}_x + w\sum_{m=1}^{N-1}\left(|m+1\rangle\langle m| \otimes \frac{\hat{\sigma}_x + i\hat{\sigma}_y}{2} + h.c.\right)$$

Hamiltonian of a long piece: if w<v, 0-energy states at left/right ends, on A/B sublattice



$$H = \begin{pmatrix} 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \end{pmatrix}$$



Bulk Hamiltonian: describe in momentum space. Periodic boundaries, Fourier transform in external degrees of freedom only

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{imk} |m\rangle, \qquad \text{for } k \in \{\delta_k, 2\delta_k, \ldots, N\delta_k\} \quad \text{with } \delta_k = \frac{2\pi}{N},$$

Bulk momentum-space Hamiltonian: 2x2

$$\hat{H}(k) = \langle k| \hat{H}_{\text{bulk}} |k\rangle = \sum_{\alpha,\beta \in \{A,B\}} \langle k, \alpha| H_{\text{bulk}} |k, \beta\rangle \cdot |\alpha\rangle \langle \beta| .$$

Eigenstates: u_n(k) is 2-component, Psi_n(k) is full Bloch state

$$\hat{H}(k) |u_n(k)\rangle = E_n(k) |u_n(k)\rangle ; \qquad |u_n(k)\rangle = a_n(k) |A\rangle + b_n(k) |B\rangle$$

$$|\Psi_n(k)\rangle = |k\rangle \otimes |u_n(k)\rangle$$

Ensures H(k) periodic in momentum space → simple formulas
(need to neglect different position of sites in unit cell)

$$\hat{H}(k+2\pi) = \hat{H}(k); \qquad |\langle u_n(k)|u_n(k+2\pi)\rangle| = 1.$$

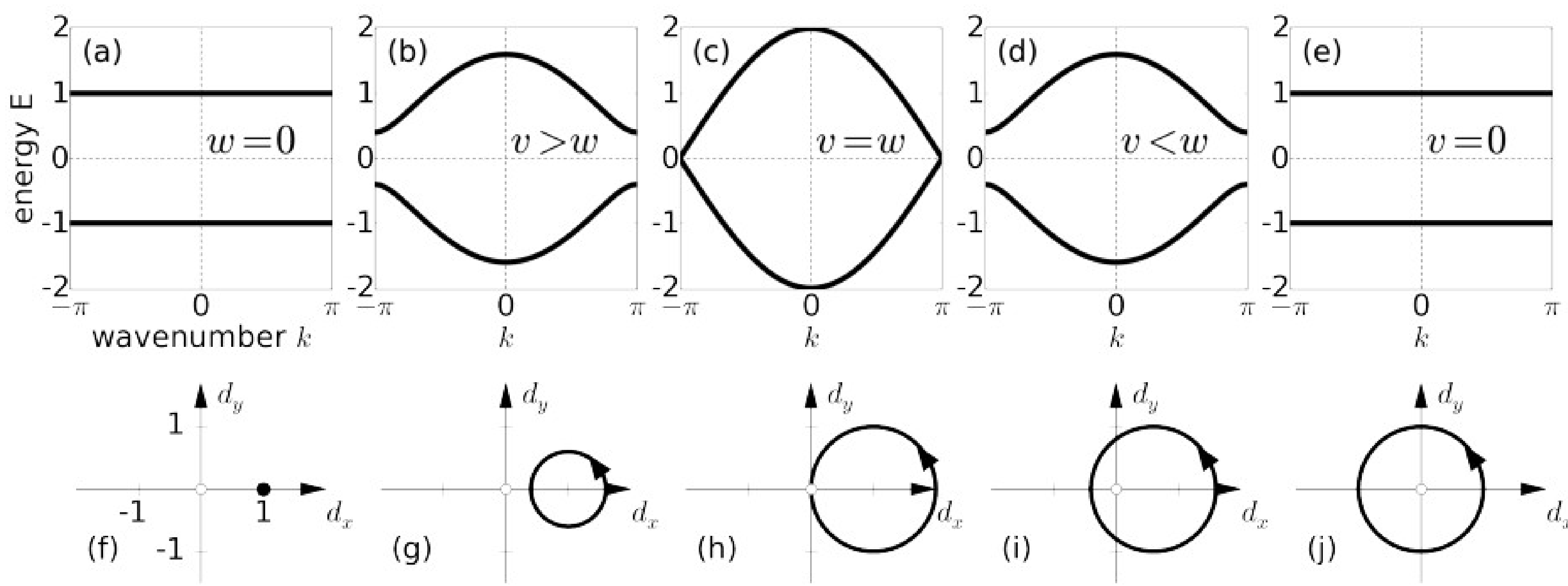
Bulk Hamiltonian of SSH: Dispersion relation has gap if v\neq w

$$H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix}$$

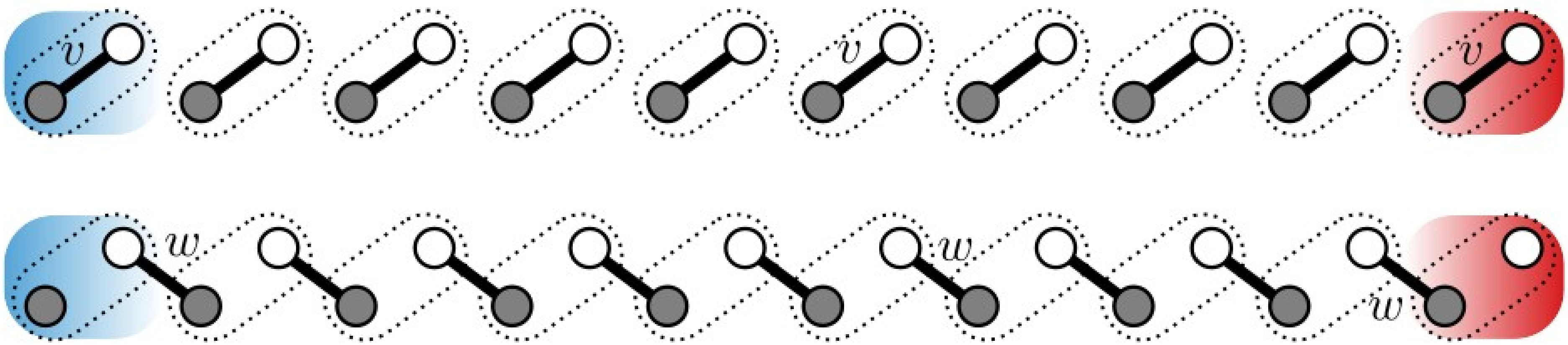
$$E(k) = \pm \left| v + e^{-ik} w \right| = \pm \sqrt{v^2 + w^2 + 2vw\cos k}.$$

$$\hat{H}(k) = d_0(k)\hat{\sigma}_0 + d_x(k)\hat{\sigma}_x + d_y(k)\hat{\sigma}_y + d_z(k)\hat{\sigma}_z = d_0(k)\hat{\sigma}_0 + \mathbf{d}(k)\hat{\sigma}.$$

$$d_x(k) = v + w\cos k; \qquad d_y(k) = w\sin k; \qquad d_0(k) = d_z(k) = 0$$

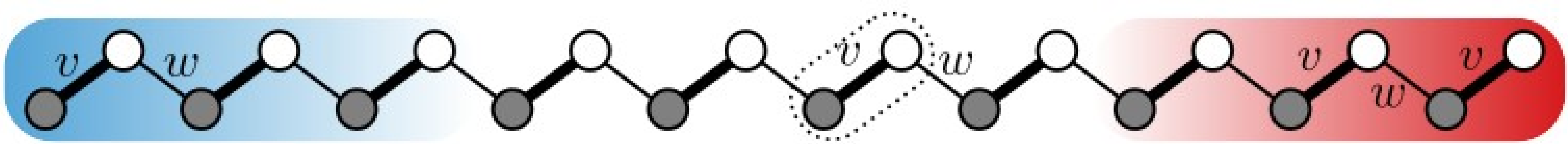


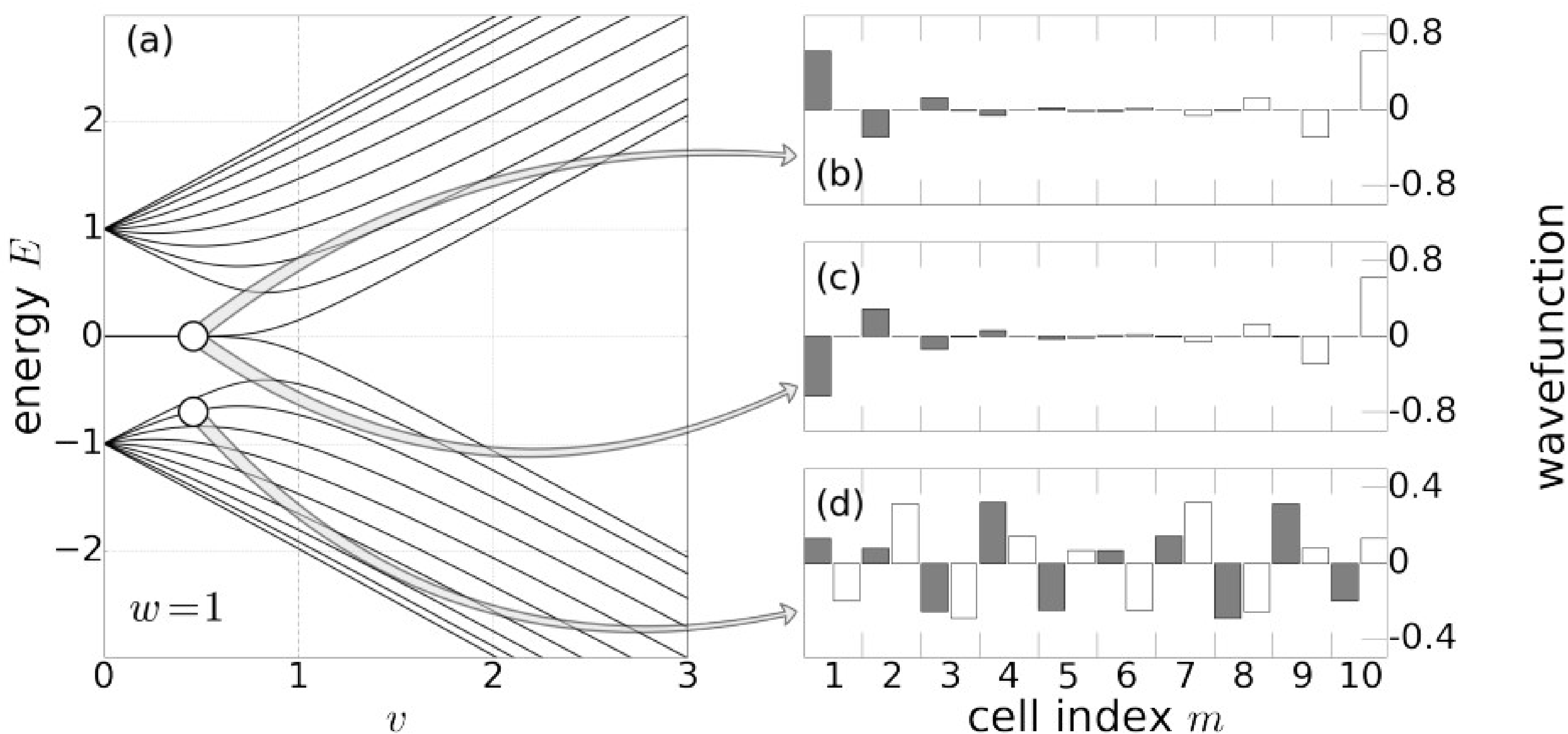
Edge states: states that have low overlap with bulk states, can have energy in bulk gap.
1) Fully dimerized limit



Fully dimerized limits: w=0 → no edge states
v=0 → 1 edge state on left edge, on sublattice A, 1 on right edge, B

Edge states: states that have low overlap with bulk states, can have energy in bulk gap.
2) Adiabatic deformation away from fully dimerized limit





The essential symmetry here is sublattice (chiral) symmetry: Only terms in the Hamiltonian connect sites of different sublattices.

Unitary symmetries: Hamiltonian block diagonal → don't care, eliminate by treating blocks separately.

$$\hat{U}\hat{H}\hat{U}^\dagger = \hat{H}$$

Chiral symmetry: No matrix element connecting sites on the same sublattice

$$\hat{H} = \sum_{m,m'} v_{m,m'} |m, A\rangle\langle m', B| + h.c. \quad \Leftrightarrow \quad \hat{\Gamma}\hat{H}\hat{\Gamma} = -\hat{H}$$

$$\hat{\Gamma} = \hat{P}_A - \hat{P}_B = \hat{\Gamma}^{-1} = \hat{\Gamma}^\dagger$$

$$\hat{P}_A = \sum_m |m, A\rangle\langle m, A|$$

Chiral symmetry: Hamiltonian block off-diagonal in chiral basis

$$\text{chiral basis:} \quad \hat{\Gamma} = \text{diag}(\underbrace{1, \ldots, 1}_{N_A}, \underbrace{-1, \ldots, -1}_{N_B}),$$

$$\hat{H} = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}$$

Essential consequence of chiral symmetry: symmetry of spectrum

→ Any state at E has a partner at -E

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \quad \Longrightarrow \quad \hat{H}\hat{\Gamma}\,|\psi\rangle = -\hat{\Gamma}\hat{H}\,|\psi\rangle = -\hat{\Gamma}E\,|\psi\rangle = -E\hat{\Gamma}\,|\psi\rangle\,.$$

→ Nonzero energy states have equal support on A and B

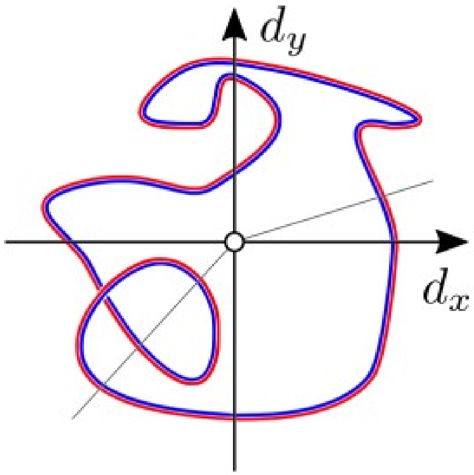
$$\text{If } E \neq 0: \quad \langle\psi|\,\hat{P}_A\,|\psi\rangle = \langle\psi|\,\hat{P}_B\,|\psi\rangle$$

→ Zero energy states are either on A or on B

$$\text{If } E = 0: \quad \hat{H}\hat{P}_{A/B}\,|\psi\rangle = \tfrac{1}{2}\hat{H}\left(|\psi\rangle \pm \hat{\Gamma}\,|\psi\rangle\right) = 0.$$

Chiral symmetry and bulk gap
→ Winding number of bulk topological invariant

$$H(k) = \begin{pmatrix} 0 & h(k) \\ h^\dagger(k) & 0 \end{pmatrix} \qquad \det h(k) = d_x(k) + i d_y(k)$$



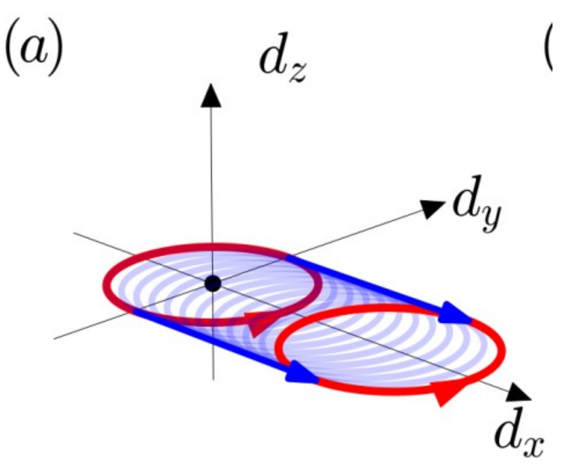
If H(k) gapped, h(k) can have no 0 eigenvalue
→ winding number makes sense

$$\nu = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \mathrm{d}k \frac{\mathrm{d}}{\mathrm{d}k} \log \det h(k)$$

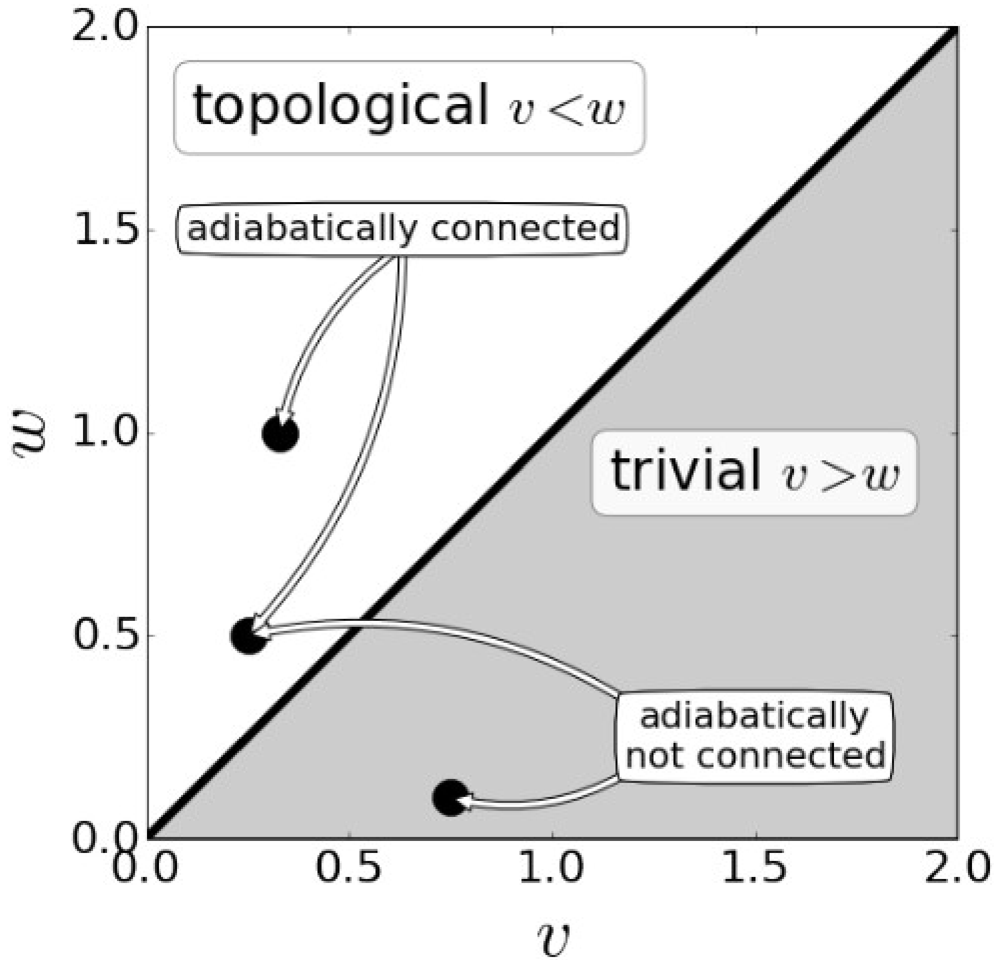
For SSH model, h(k) is 1x1 matrix, “det” not needed.
dx, dy are Pauli matrix coefficients.

Chiral symmetry == $d_z(k)=d_0(k)=0$

$$d_0(k)\hat{\sigma}_0 + \mathbf{d}(k)\hat{\sigma}.$$



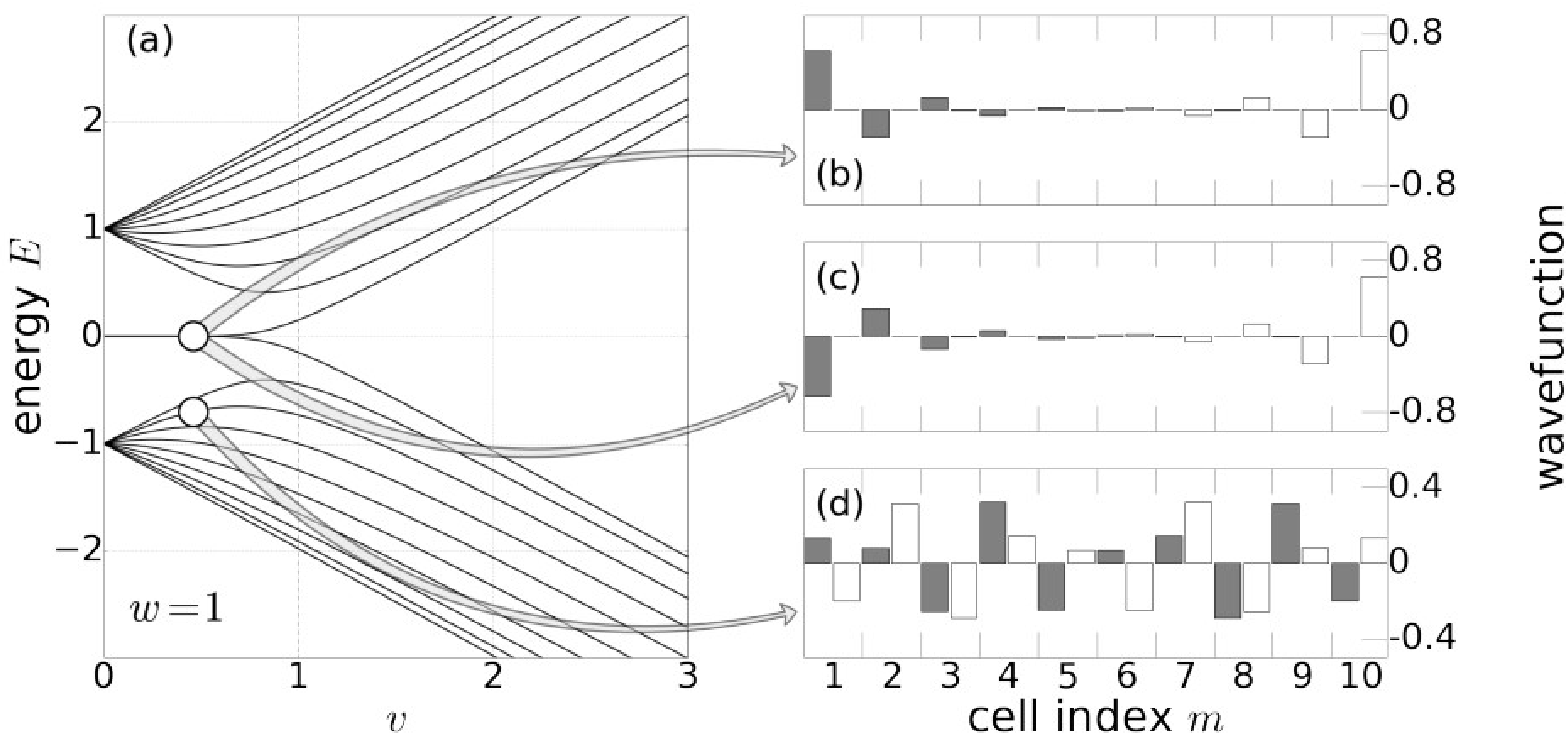
Topological phases of SSH: Parameter regions that are adiabatically connected, can be labeled by the invariant



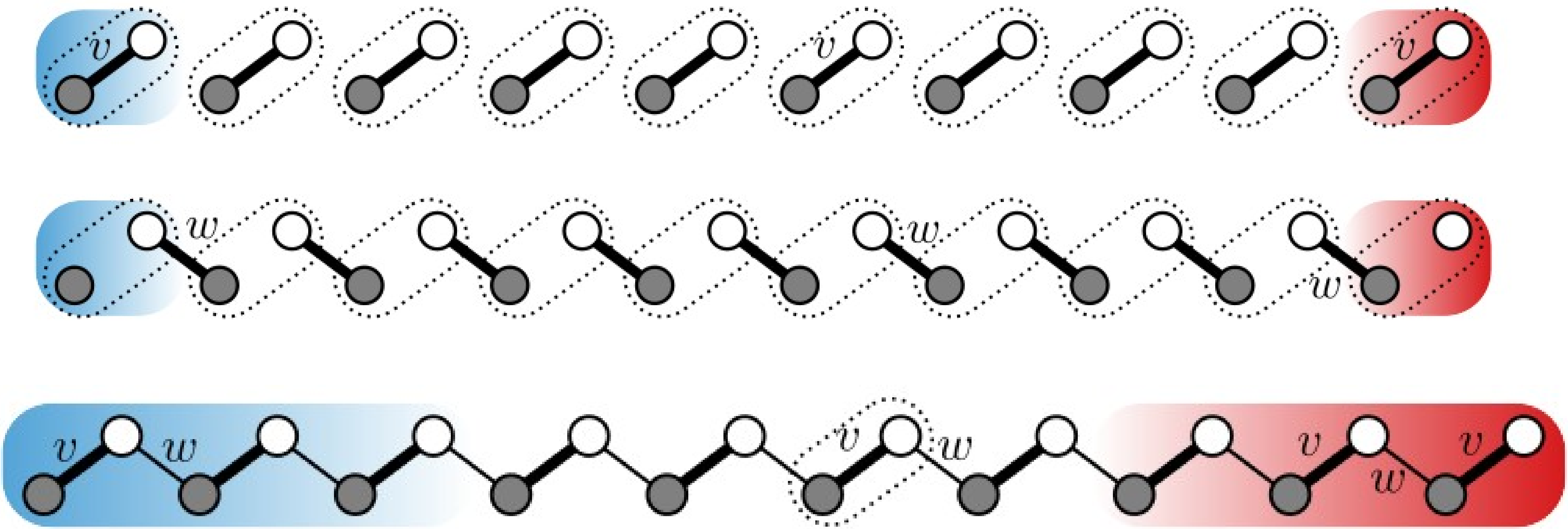
Net number of edge states is a topological invariant
Chiral symmetry local → pins edge states to 0 energy
(exponential precision, $E=e^{-L/\xi}$)

Z = #(edge states at left edge on sublattice A)
- #(edge states at left edge on sublattice B)

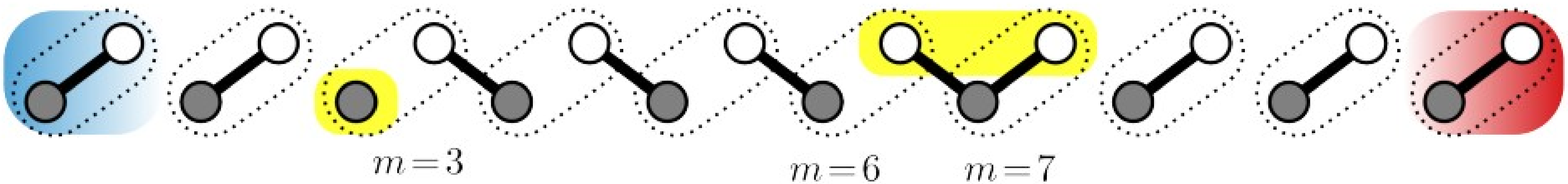
This is a topological invariant: invariant under adiabatic deformations:
- continuous changes
- during which bulk gap does not close
- during which symmetries are intact



Bulk-boundary correspondence: ν=Z
Bulk winding number = net number of edge states
proof: here by adiabatic deformation

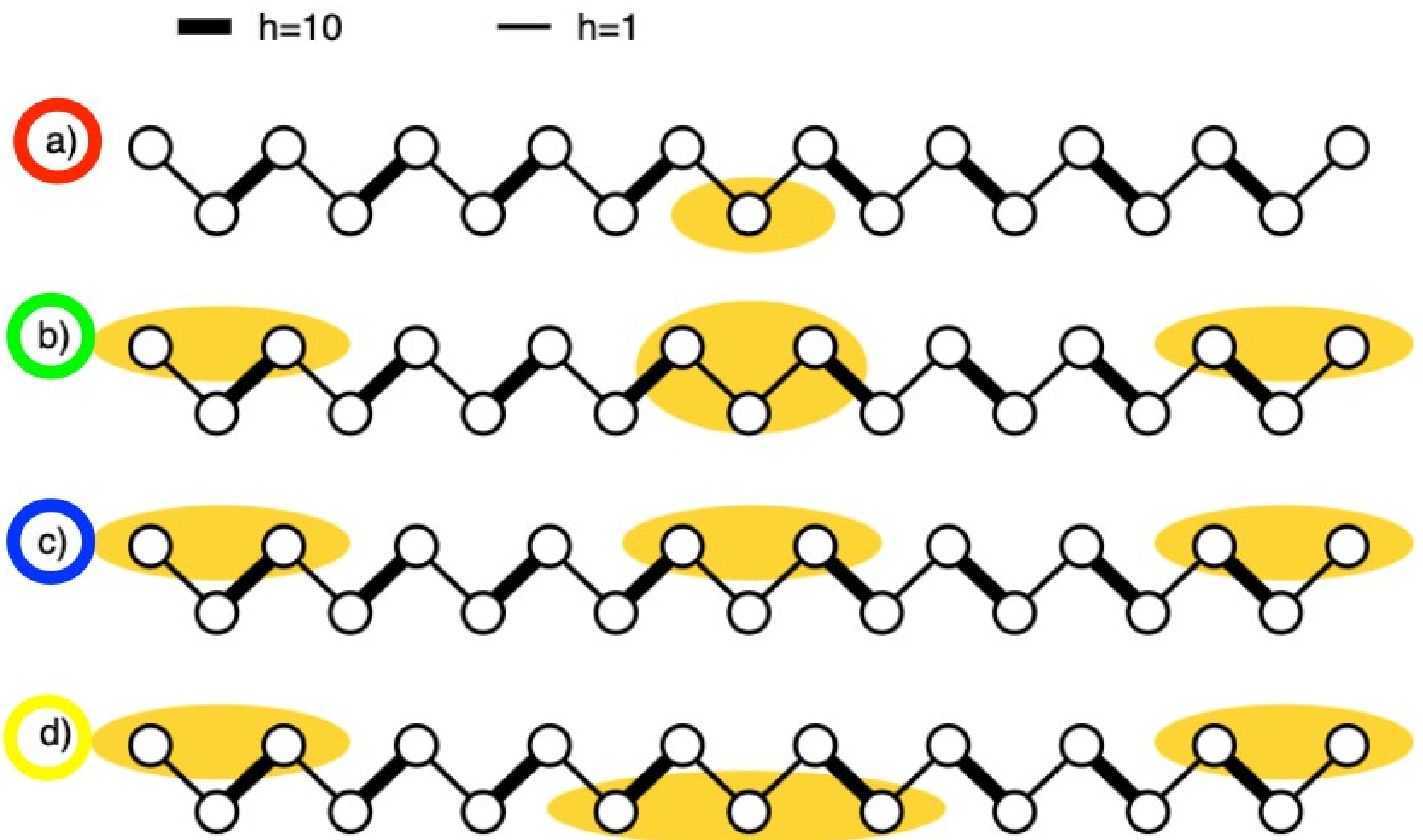


Consequence of Bulk-Boundary correspondence: Topologically protected states at an interface between two bulks, $Z = \nu_1 - \nu_2$



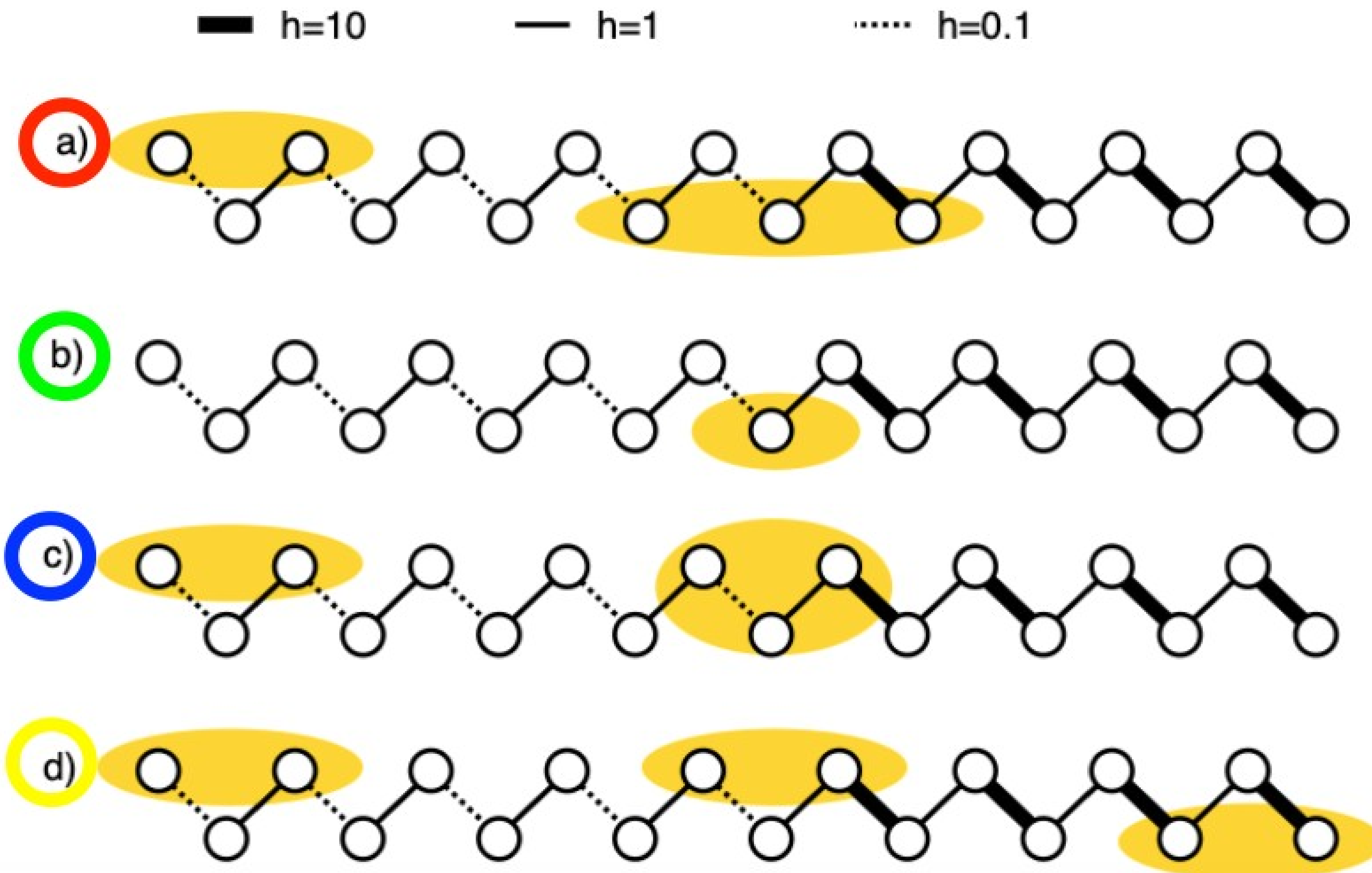
Where are the edge states?

An SSH chain is depicted below, with 2 types of hopping amplitudes h (different line styles). Where are the edge states? The shading indicates sites
over which the edge state wavefunctions extend. Please choose the best answer.



Where are the edge states? 1

An SSH chain is depicted below, with 3 types of hopping amplitudes h (different line styles). Where are the zero-energy bound states? The shading indicates sites over which the wavefunctions extend. Please choose the best answer.



SSH glued

We take two SSH chains with open ends. We glue them into a circle. As a result, the total number of zero-energy bound states...

a) cannot change

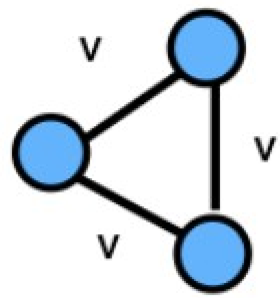
b) can increase

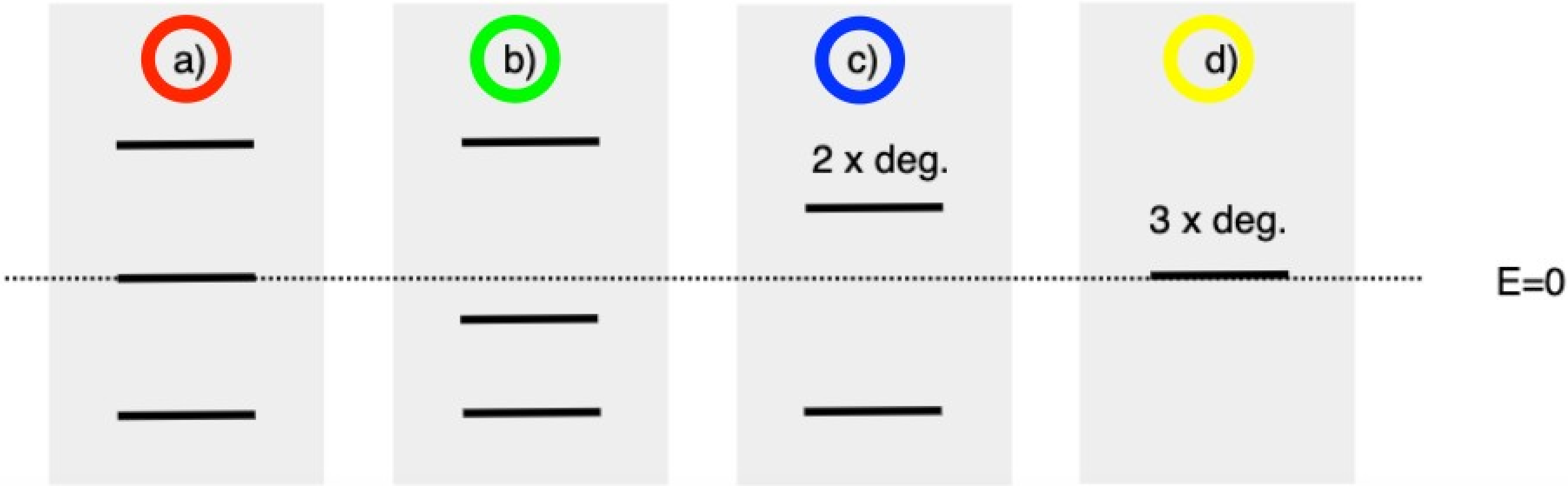
c) can decrease

d) can increase or decrease

3-site ring

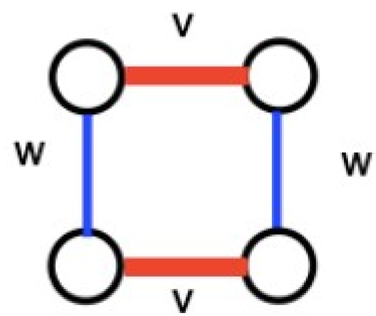
Which is the spectrum of a 3-site ring?
No onsite potentials, just real valued nearest neighbor hopping.
Dotted line = zero of energy



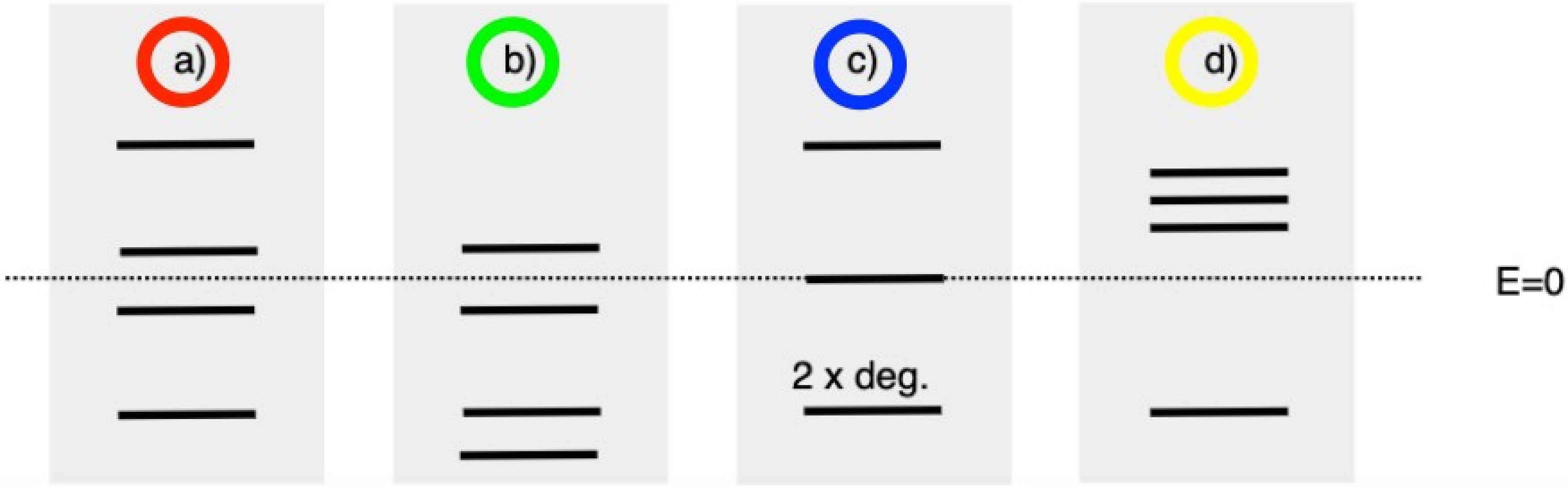


4-site ring 1.

Which is the spectrum of the 4-site ring?
No onsite potential, just nearest neighbor hopping.
Dotted line = zero of energy.

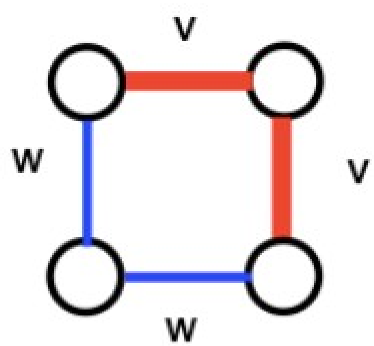


$v \neq w \qquad v, w > 0$

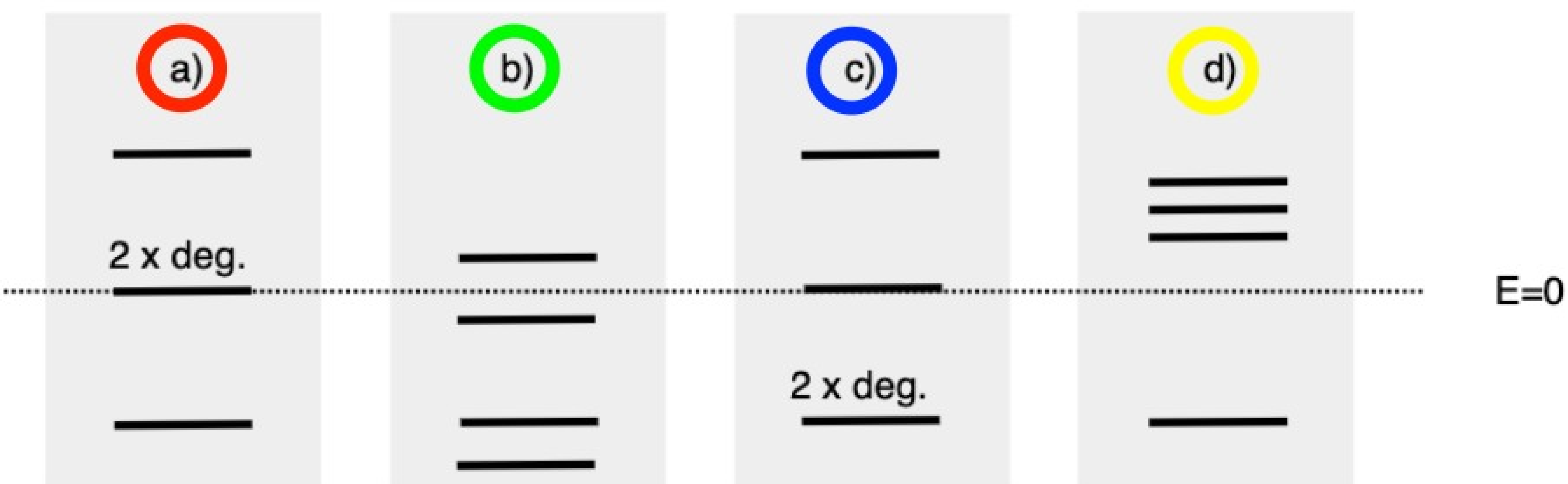


4-site ring 2.

Which is the spectrum of the 4-site ring?
No onsite potential, just nearest neighbor hopping.
Dotted line = zero of energy.



$v \neq w \qquad v, w > 0$



Zero modes and chiral symmetry 1.

For zero-energy eigenstates of a single-particle Hamiltonian with chiral symmetry, it is true that…

a) they are chiral symmetric partners of themselves.

b) they break chiral symmetry.

c) their total number is always odd.

d) they can be chiral symmetric partners of themselves.

Zero modes and chiral symmetry 2.

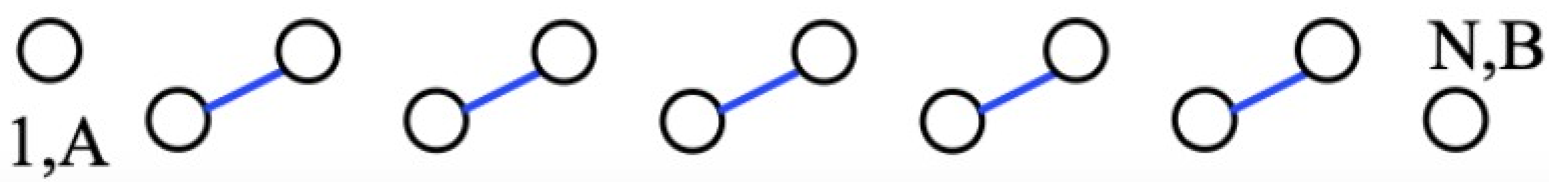
Take the topological, completely dimerized limit of the SSH model.
What is true for the state $|1,A\rangle + |N,B\rangle$?

a) Since this is a linear combination of two eigenstates, its energy is nonzero.

b) Since this is a zero energy eigenstate, it is its own chiral symmetric partner.

c) This is a zero energy eigenstate, which is not its own chiral symmetric partner.

d) Since the system is of finite length, this linear combination of states has positive energy.



Zero modes and chiral symmetry 3.

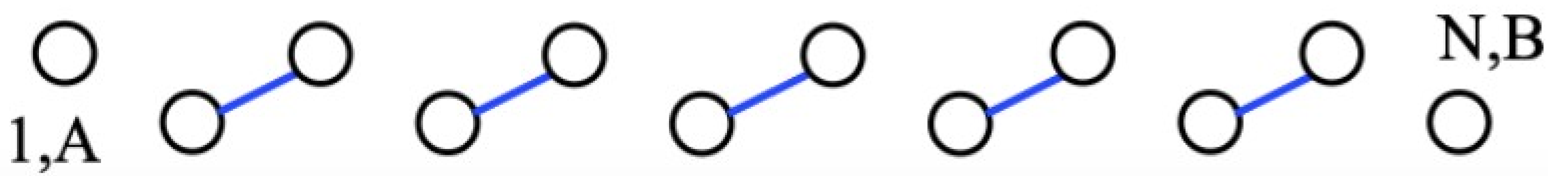
Take the topological, completely dimerized limit of the SSH model, with v=0 and w=1.
What is true for the energy eigenstates |1,B⟩+|2,A⟩ and |2,B⟩-|3,A⟩ ?

a) They are eigenstates of the chiral symmetry operator.

b) They are transformed into each other by the chiral symmetry operator.

c) Since they are both energy eigenstates, their arbitrary linear combination is also an energy eigenstate

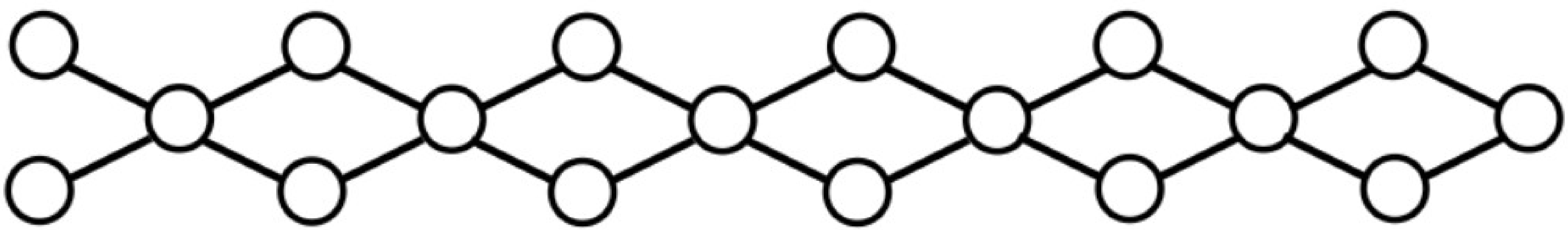
d) They are eigenstates at opposite energy, however, they are not chiral symmetric partners of each other.



Quasi-one-dimensional system 1.

We add an extra row of atoms to the SSH model
(3 rows of atoms, ABA stacking, only diagonal hoppings).
No onsite energies, only hopping.

Does this Hamiltonian have chiral symmetry?



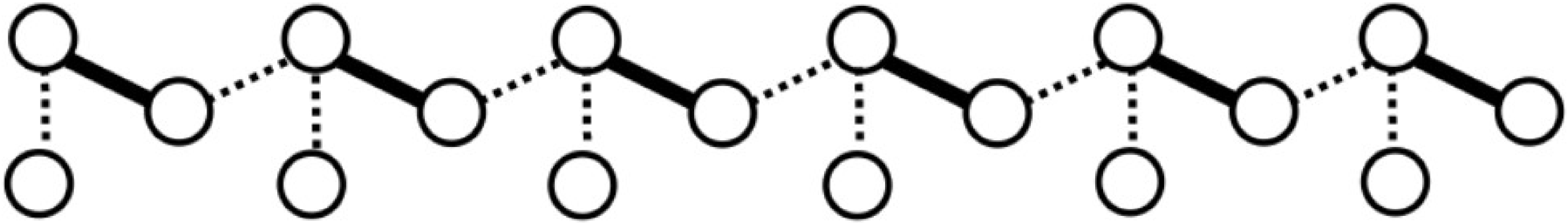
a) yes (what operator represents it?)

b) no (why?)

c) cannot decide (what does it depend on?)

Quasi-one-dimensional system 2.

We add an extra row of atoms to the SSH model. No onsite energies, only hopping.
Thick lines: hopping = 10. Thin dotted lines: hopping = 1.



Does this Hamiltonian have chiral symmetry?

a) yes (what operator represents it?)

b) no (why?)

c) cannot decide (what does it depend on?)

Complex hopping 2.

In the SSH model with real valued nearest neighbor hopping there are two topological classes. How many topological classes do we have if we allow for a real valued third neighbor hopping, as depicted by green and magenta lines?

a) 4

b) 1

c) 2

d) 3