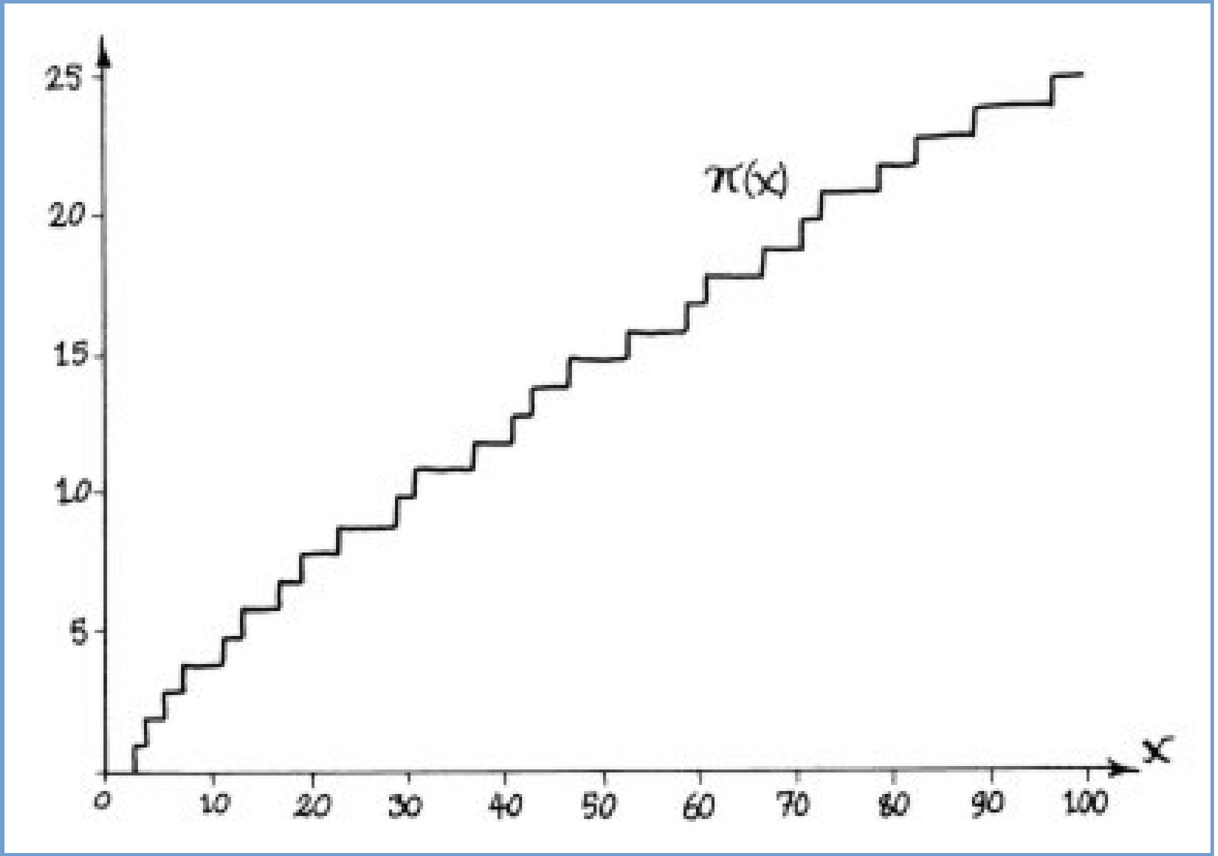
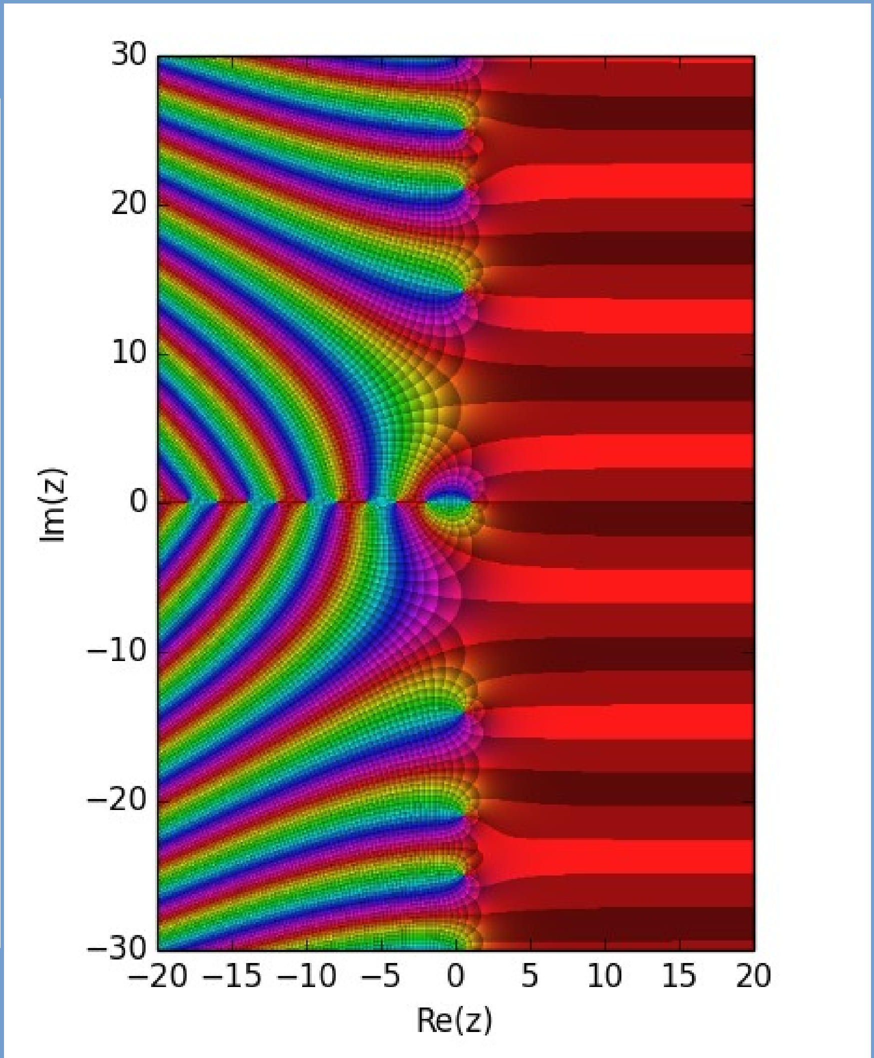
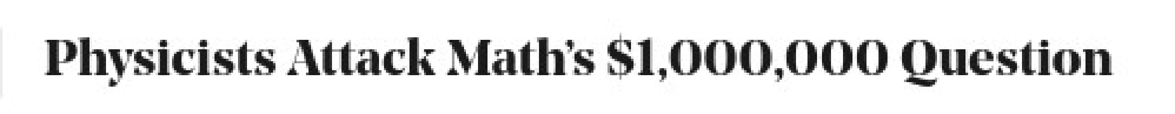
The primes, the zeta function and quantum mechanics...





...and literally everything



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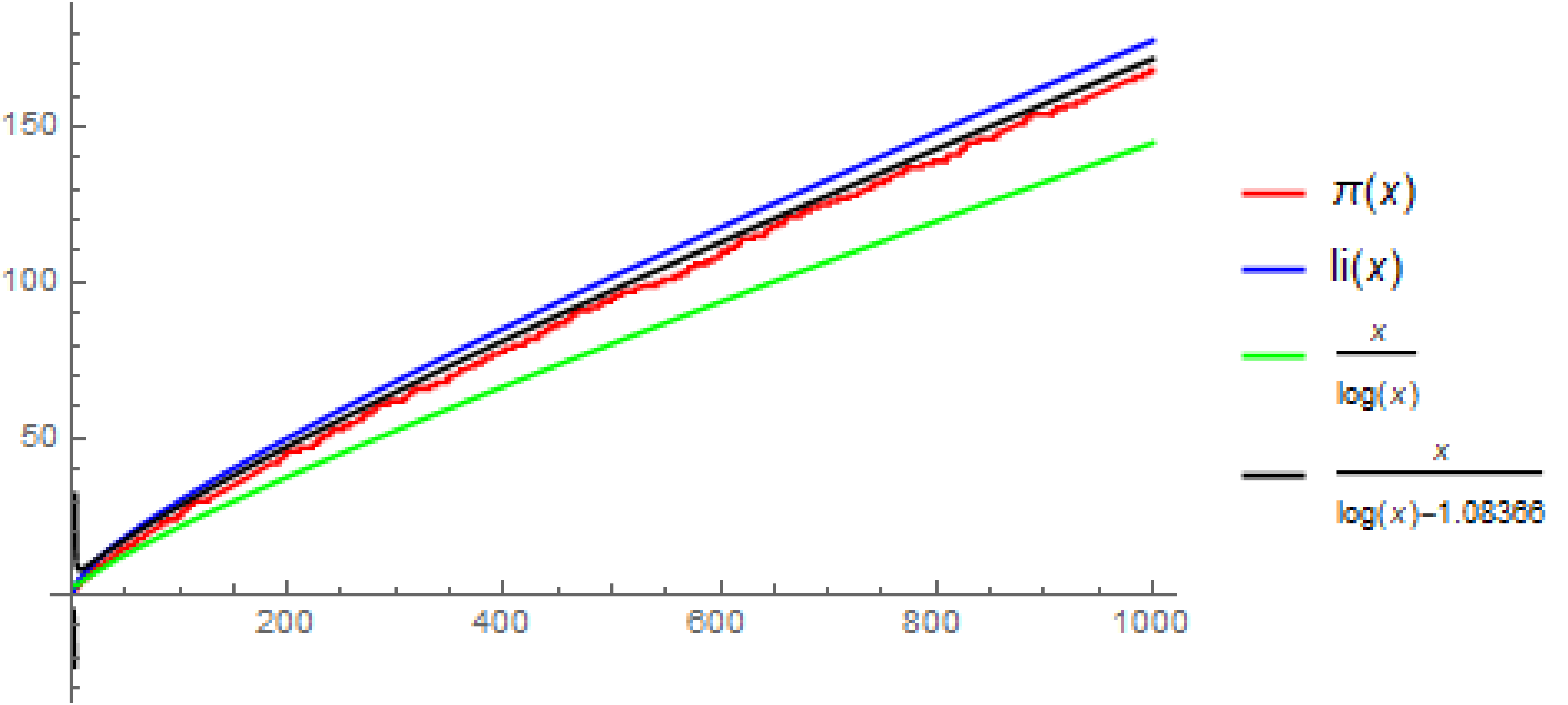
Outline

- Number theory, history of mathematics
- Formulation of the Riemann hypothesis
- Relation with QM
- Modern approaches

The number of primes

- Approximations (Gauss…)

$$Li(x)=\int_2^x \frac{1}{\ln(t)}\,dt$$



The number of primes

- Riemann gave a better approximation (1859)!
  - He gave an exact result (includes an infinite sum)!
    - He used the Riemann zeta function

$$\zeta(x) = \sum_n \frac{1}{n^x} = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \ldots$$

    - The exact formula:

$$f(x) = \mathrm{li}(x) - \sum_\rho \mathrm{li}(x^\rho) - \log(2) + \int_x^\infty \frac{dt}{t(t^2-1)\log(t)}$$

The zeta function

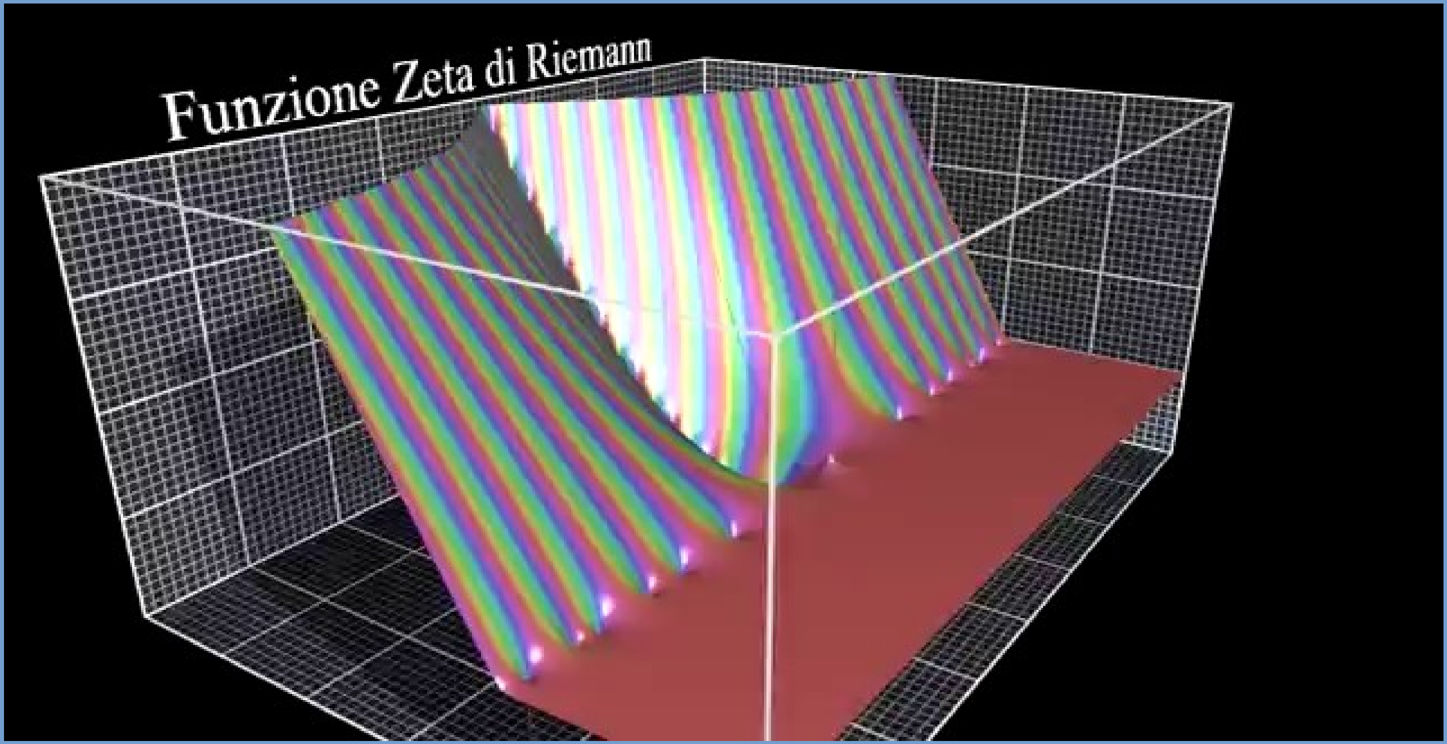
- Introduced in the 18th century $\zeta(x)=\sum_n \frac{1}{n^x}=\prod_p \sum_n \frac{1}{p^{nx}}$
- NOT defined for numbers *x* for which *Re(x)* <= *1*!
- The zeta function is holomorphic in this domain!
- => analytic continuation:

$$\zeta(z)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}}{e^x-1}\,dx$$

The zeros of the Riemann function

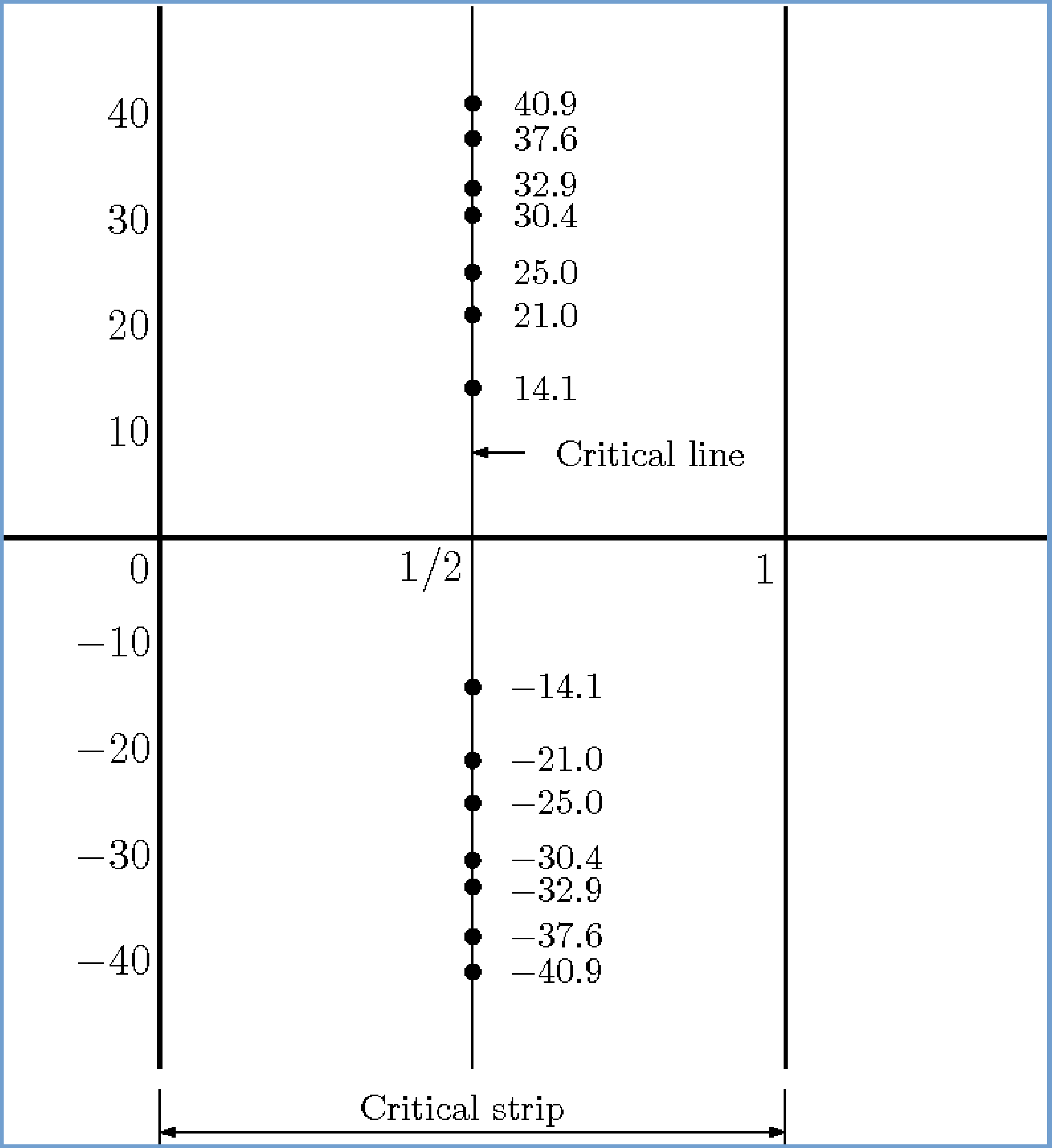
- Riemann used the zeros in his exact formula!

$\zeta(0) = -\frac{1}{2}$ $\zeta(-1) = -\frac{1}{12}$ $\zeta(-2n) = 0$



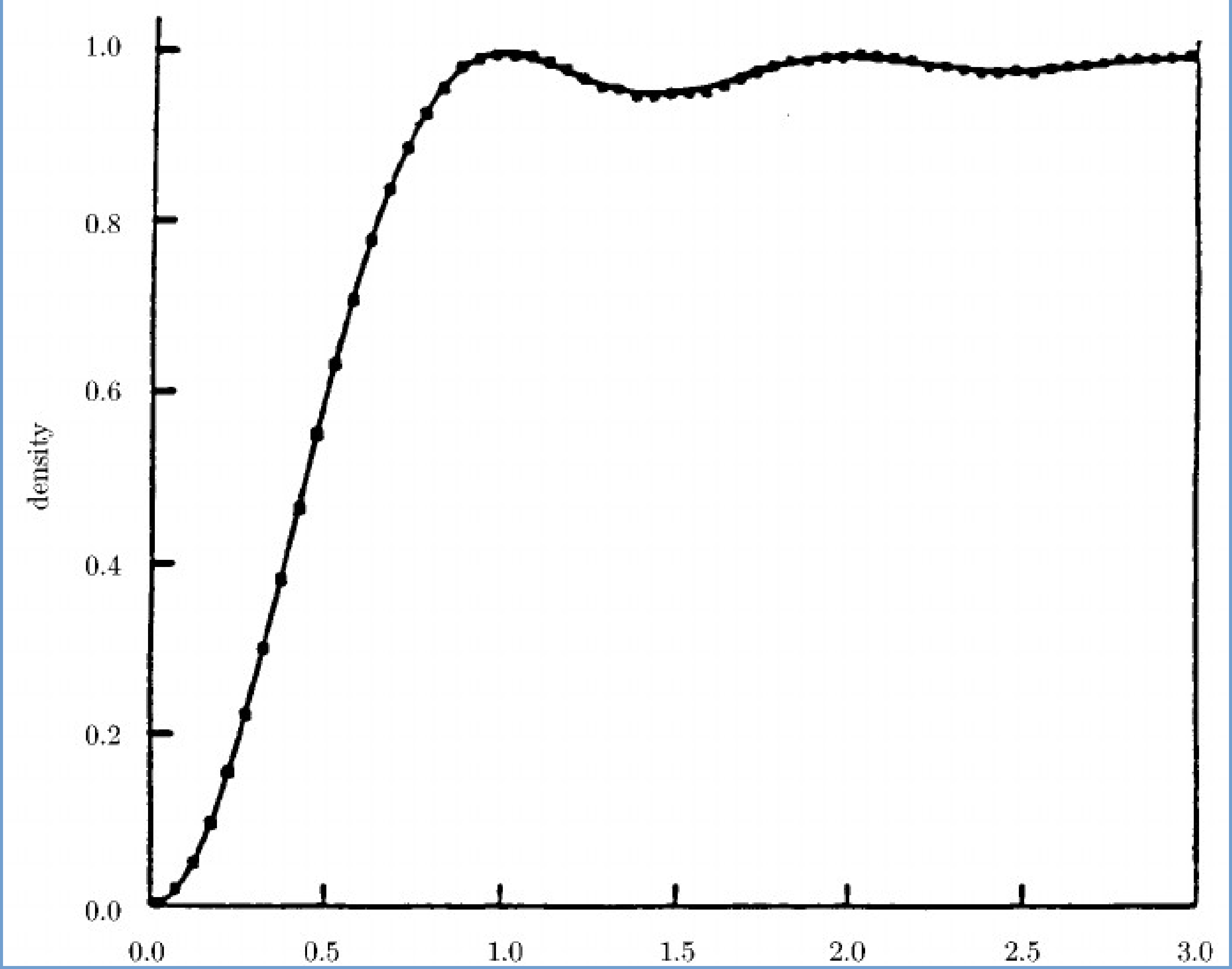
The zeros of the Riemann function

- Critical strip
- Critical line(?)
- Infinitely many zeros on the critical line! (1914 – Hardy)
- $\approx 10^{13}$ zeros found on the critical line



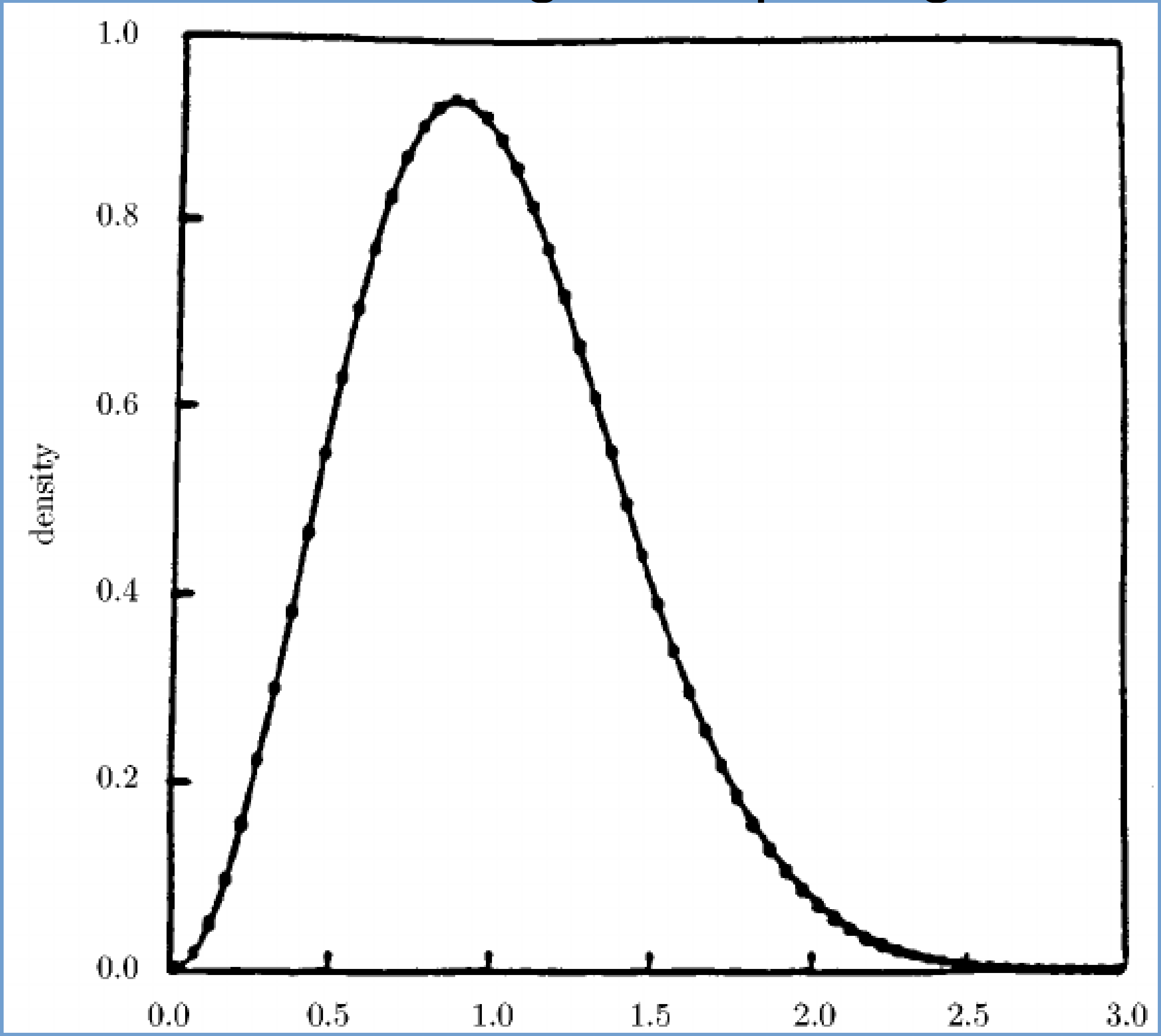
A new approach

Pair correlation of the zeros (Montgomery 1973)



A new approach

Nearest neighbor spacing



Random Matrix Theory

- Matrix with random elements
- Hamiltonians lacking time-reversal symmetry: Gaussian unitary ensemble GUE

$$density \sim e^{-\frac{n}{2}Tr\left(H^2\right)}$$

- The zeros of the Riemann function seem to have similar distribution as the eigenvalues

$$level\,spacings\left(s\right) \sim s^2 e^{-c \cdot s^2}$$

The Hilbert–Pólya conjecture

- If exists an operator of the form $\frac{1}{2} + i \cdot H$ (H is a Hamiltonian)
- And H’s eigenvalues are the real parts of the zeros of the Riemann function

  => The Riemann hypothesis is true!
- It might be easier to construct such a Hamiltonian than anything else…
- But what can the Hamiltonian be? ($\leftarrow$ 1 million $)

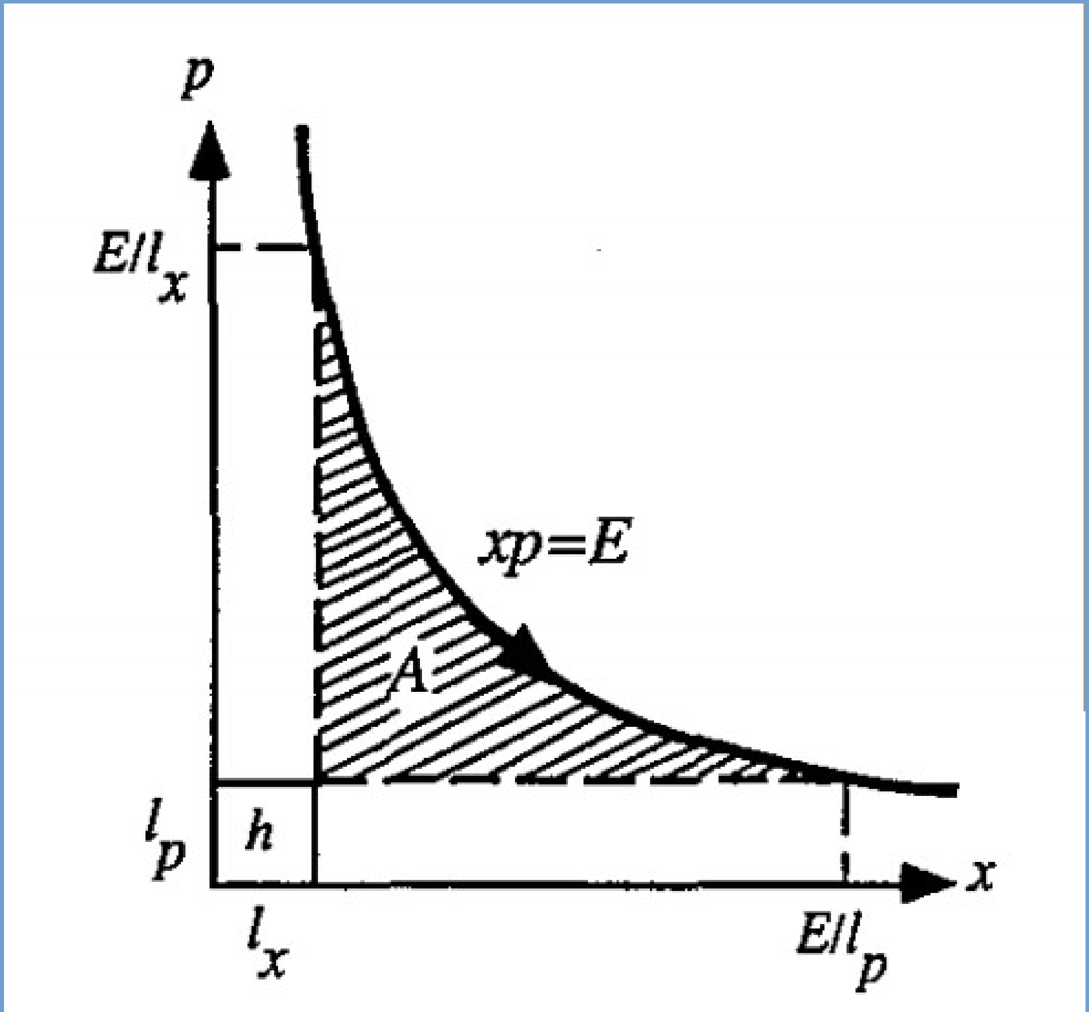
The Berry-Keating conjecture (1999)

- No time reversal symmetry
- Chaotic dynamics
- Many others...
- The classical form of the Hamiltonian might be:

$$H = x \cdot p$$

H = xp

- Semiclassical level counting



$$N(E)=\frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right)-1\right)+\frac{7}{8}+\ldots$$

- Exactly the same as the asymptotic form of the smoothed counting function for the Riemann zeros

What is $\hat{H}$?

- Simplest possibility: $$\hat{H} = \frac{1}{2}\left(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}\right)$$

- In a paper in 2017 (Bender Hamiltonian):

$$\hat{H}_B = \frac{1}{1 - e^{-i\hat{p}}}\left(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}\right)\left(1 - e^{-i\hat{p}}\right)$$

The Bender Hamiltonian

- It reduces to xp in the classical limit
- It is not hermitian, but $i\hat{H}_B$ is invariant under parity-time reflection
  - The eigenvalues are either real or occur in cc pairs
  - This is the problem…
- Modified inner product

The Bender Hamiltonian

- It satisfies the conditions of the Hilbert-Pólya conjecture
  - Eigenfunctions: Hurwitz zeta function:
    - Modified version of the Riemann zeta fn.

$$\Psi_z(x) = -\zeta(z, x+1)$$

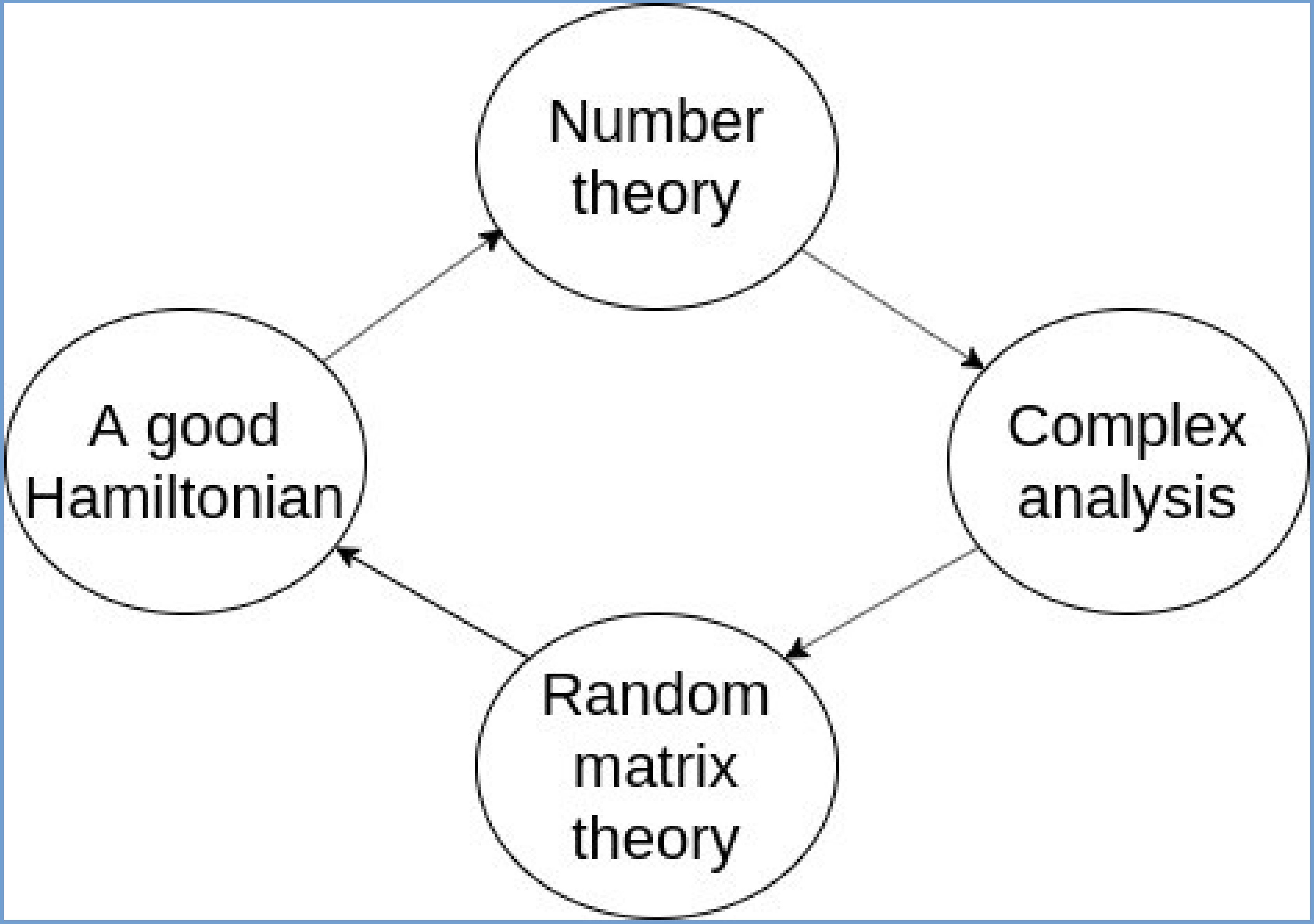
- Boundary condition: $\Psi_z(0) = 0$

The Bender Hamiltonian

Comment on ‘Comment on “Hamiltonian for the zeros of the Riemann zeta function” ’

- Problems:
  - Hilbert space is not properly defined
  - Self-adjointness of the momentum operator
  - Eigenfunctions may not be integrable

Summary


Number theory
Complex analysis
Random matrix theory
A good Hamiltonian

Questions Worth Asking

Question 1

Q: If the Riemann hypothesis was proven, what impact would it have on our cryptography (RSA)?

A: Probably no impact AT ALL.

Question 2

Q: Is this the only method people are trying to attack the Riemann hypothesis with?

A: Not at all! There exist for example methods based on:

- Geometry
- Complex analysis (results based on von Neumann and others)
  - “simple” proof...

Question 3

Q: Is it possible that the Riemann hypothesis is undecidable?

A: Yes… and this makes it true!