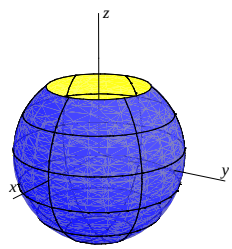
Stokes' Theorem

1. *Let $\vec{F}(x,y,z) = \langle -y, x, xyz\rangle$ and $\vec{G} = \operatorname{curl}\vec{F}$. Let $\mathcal{S}$ be the part of the sphere $x^2+y^2+z^2 = 25$ that lies below the plane $z = 4$, oriented so that the unit normal vector at $(0,0,-5)$ is $\langle 0,0,-1\rangle$. Use Stokes' Theorem to find $\displaystyle\iint_{\mathcal{S}} \vec{G}\cdot d\vec{S}$.*

   Solution. Here's a picture of the surface $\mathcal{S}$.



   To use Stokes' Theorem, we need to first find the boundary C of $\mathcal{S}$ and figure out how it should be oriented. The boundary is where $x^2+y^2+z^2 = 25$ and $z = 4$. Substituting $z = 4$ into the first equation, we can also describe the boundary as where $x^2+y^2 = 9$ and $z = 4$.

   To figure out how C should be oriented, we first need to understand the orientation of $\mathcal{S}$. We are told that $\mathcal{S}$ is oriented so that the unit normal vector at $(0,0,-5)$ (which is the lowest point of the sphere) is $\langle 0,0,-1\rangle$ (which points down). This tells us that the blue side must be the "positive" side.

   We want to orient the boundary so that, if a penguin walks near the boundary of $\mathcal{S}$ on the "positive" side (which we've already decided is the blue side), he keeps the surface on his left. If we imagine looking down on the surface from a really high point like $(0,0,100)$, then the penguin should walk clockwise (from our vantage point).

   So, using Stokes' Theorem, we have changed the original problem into a new one:

Evaluate the line integral $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, where C is the curve described by $x^2+y^2 = 9$ and $z = 4$, oriented clockwise when viewed from above.

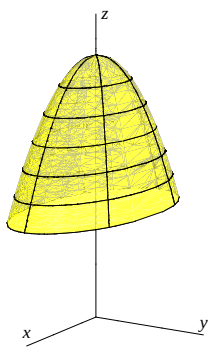
   Now, we just need to evaluate the line integral, using the definition of the line integral. (This is like #4(a) on the worksheet "Vector Fields and Line Integrals".) We start by parameterizing C. One possible parameterization is $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4\rangle$, $0 \le t < 2\pi$.[1] If we look at this from above, it is oriented counterclockwise, which is the wrong orientation. Therefore,

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= -\int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\ dt \\
&= -\int_0^{2\pi} \langle -3\sin t, 3\cos t, 36\cos t\sin t\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle\ dt \\
&= -\int_0^{2\pi} 9\ dt \\
&= \boxed{-18\pi}
\end{aligned}$$

[1] To come up with this, remember that we can parameterize a circle $x^2+y^2 = 1$ in $\mathbb{R}^2$ by $x = \cos t$, $y = \sin t$ (and, as t increases, this goes around the circle counterclockwise). Here, we're looking at $x^2+y^2 = 9$; if we rewrite this as $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, then we can write $\frac{x}{3} = \cos t$, $\frac{y}{3} = \sin t$.

2. *Let $\vec{F}(x,y,z) = \langle -y, x, z\rangle$. Let $\mathcal{S}$ be the part of the paraboloid $z = 7 - x^2 - 4y^2$ that lies above the plane $z = 3$, oriented with upward pointing normals. Use Stokes' Theorem to find* $\displaystyle\iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$.

Solution. Here is a picture of the surface $\mathcal{S}$.



The strategy is exactly the same as in #1. The boundary is where $z = 7 - x^2 - 4y^2$ and $z = 3$, which is the same as $x^2 + 4y^2 = 4$ and $z = 3$.

Since $\mathcal{S}$ is oriented with normals pointing upward, the top side of the paraboloid (the yellow side in the picture) is the "positive" side. If we imagine looking down on the surface from above, then a penguin walking around on the "positive" (yellow, in this case) side keeps the surface on his left by walking counterclockwise.

Therefore, by Stokes' Theorem, the original problem can be rewritten as

Evaluate the line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where C is the curve described by $x^2 + 4y^2 = 4$ and $z = 3$, oriented counterclockwise when viewed from above.

A parameterization of this curve is $\vec{r}(t) = \langle 2\cos t, \sin t, 3\rangle$.[(2)] This goes counterclockwise when viewed from above (as we want), so

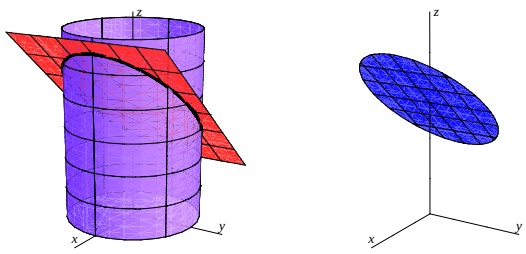
$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_0^{2\pi} \langle -\sin t, 2\cos t, 3\rangle \cdot \langle -2\sin t, \cos t, 0\rangle\ dt \\
&= \int_0^{2\pi} (2\sin^2 t + 2\cos^2 t)\ dt \\
&= \int_0^{2\pi} 2\ dt \\
&= \boxed{4\pi}
\end{aligned}$$

[(2)] To come up with this parameterization, rewrite $x^2 + 4y^2 = 4$ as $\left(\frac{x}{2}\right)^2 + y^2 = 1$ and then use $\frac{x}{2} = \cos t$, $y = \sin t$. It's easy to check that it's reasonable: if we plug in $x = 2\cos t$, $y = \sin t$, and $z = 3$, then the equations $x^2 + 4y^2 = 4$ and $z = 3$ are indeed satisfied.

3. *The plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$ intersect in a curve C. Suppose C is oriented counterclockwise when viewed from above. Let $\vec{F}(x, y, z) = \langle x^3 + 2y, \sin y + z, x + \sin z^2 \rangle$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. We'll use Stokes' Theorem. To do this, we need to think of an oriented surface $\mathcal{S}$ whose (oriented) boundary is C (that is, we need to think of a surface $\mathcal{S}$ and orient it so that the given orientation of C matches). Then, Stokes' Theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$. Let's compute $\operatorname{curl} \vec{F}$ first. (It's worthwhile to do this first because, if we find out it's $\vec{0}$, then we know the integral will be 0 without any more work.) In this case, $\operatorname{curl} \vec{F} = \langle -1, -1, -2 \rangle$.

Now, let's think of a surface whose boundary is the given curve C. We are told that C is the intersection of a plane and a cylinder (left picture), so one surface we could use is the part of the plane inside the cylinder (right picture):



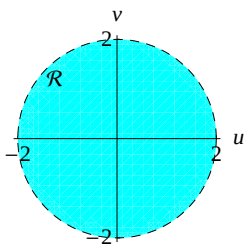
Let's call this $\mathcal{S}$ and figure out how it should be oriented. We want to orient $\mathcal{S}$ so that, if a penguin walks along the given curve C (going counterclockwise when viewed from above) on the "positive" side of $\mathcal{S}$, he keeps the surface on his left. This means that we want the top side of $\mathcal{S}$ to be the "positive" side, so we should orient $\mathcal{S}$ with normals pointing upward.

So, using Stokes' Theorem, we have changed the original problem into a new one:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the plane $z = x + 4$ inside the cylinder $x^2 + y^2 = 4$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \langle -1, -1, -2 \rangle$.

To do this new problem, let's follow the same three steps we used in #4(a) on the worksheet "Flux Integrals".

First, we parameterize $\mathcal{S}$. Since the plane has equation $z = x+4$, we can use x and y as our parameters. If we let $x = u$ and $y = v$, then $z = u+4$. This gives the parameterization $\vec{r}(u, v) = \langle u, v, u+4 \rangle$. Since we are only interested in the part of the plane inside the cylinder $x^2 + y^2 = 4$, we want $x^2 + y^2 < 4$. In terms of u and v, this says $u^2 + v^2 < 4$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is a disk:



Next, we need to see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, 1 \rangle \\ \vec{r}_v &= \langle 0, 1, 0 \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -1, 0, 1 \rangle \end{aligned}$$

This always points upward, which matches the orientation we want. So, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle -1, -1, -2 \rangle \cdot \langle -1, 0, 1 \rangle\ dA \\ &= \iint_{\mathcal{R}} -1\ dA \end{aligned}$$

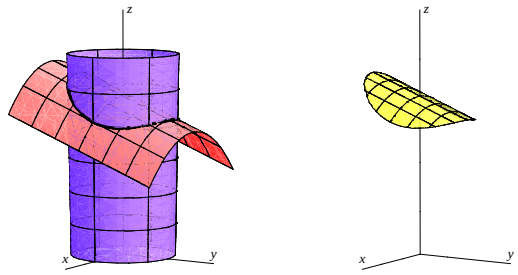
Although we could evaluate this double integral by converting it to an iterated integral, there is an easier way — remember that $\iint_{\mathcal{R}} 1\ dA$ gives the area of $\mathcal{R}$ (see #2(a) on the worksheet "Double Integrals"). Therefore,

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= -\iint_{\mathcal{R}} 1\ dA \\ &= -(\text{area of } \mathcal{R}) \\ &= \boxed{-4\pi} \end{aligned}$$

4. *Let C be the oriented curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, 8 - \cos^2 t - \sin t \rangle$, $0 \le t < 2\pi$, and let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = \left\langle z^2 - y^2, -2xy^2, e^{\sqrt{z}} \cos z \right\rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. The line integral is very difficult to compute directly, so we'll use Stokes' Theorem. The curl of the given vector field $\vec{F}$ is $\operatorname{curl} \vec{F} = \langle 0, 2z, 2y - 2y^2 \rangle$.

To use Stokes' Theorem, we need to think of a surface whose boundary is the given curve C. First, let's try to understand C a little better. We are given a parameterization $\vec{r}(t)$ of C. In this parameterization, $x = \cos t$, $y = \sin t$, and $z = 8 - \cos^2 t - \sin t$. So, we can see that $x^2 + y^2 = 1$ and $z = 8 - x^2 - y$. In other words, C must be the intersection of the surface $x^2 + y^2 = 1$ (which is a cylinder) and the surface $z = 8 - x^2 - y$ (which we don't need to visualize particularly well, beyond noticing that it's the graph of a function $f(x,y) = 8 - x^2 - y$). So, one surface we could use is the part of the surface $z = 8 - x^2 - y$ inside the cylinder $x^2 + y^2 = 1$ (right picture).



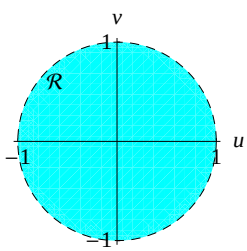
Let's call this surface $\mathcal{S}$ and figure out how it should be oriented. The original curve was parameterized using $x = \cos t$, $y = \sin t$, so when viewed from above, it was oriented counterclockwise. Therefore, we want to orient $\mathcal{S}$ so that is top is the "positive" side (a penguin walking on the top along the boundary, going counterclockwise when viewed from above, keeps the surface on his left). So, we should orient $\mathcal{S}$ with normals pointing upward.

So, using Stokes' Theorem, we have changed the original problem into a new one:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the surface $z = 8 - x^2 - y$ inside the cylinder $x^2 + y^2 = 1$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \langle 0, 2z, 2y - 2y^2 \rangle$.

To do this new problem, let's follow the same three steps we used in #4(a) on the worksheet "Flux Integrals".

First, we parameterize $\mathcal{S}$. Since the surface has equation $z = 8 - x^2 - y$, we can parameterize it as $\vec{r}(u, v) = \langle u, v, 8 - u^2 - v \rangle$. Since we are only interested in the part of the surface inside the cylinder $x^2 + y^2 = 1$, we want $x^2 + y^2 < 1$; in terms of u and v, this says $u^2 + v^2 < 1$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is a disk:



Next, we need to see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, -2u \rangle \\ \vec{r}_v &= \langle 0, 1, -1 \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 2u, 1, 1 \rangle \end{aligned}$$

This always points upward, which matches the orientation we want. So, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA \\ &= \iint_{\mathcal{R}} \langle 0, 2(8 - u^2 - v), 2v - 2v^2 \rangle \cdot \langle 2u, 1, 1 \rangle \, dA \\ &= \iint_{\mathcal{R}} (16 - 2u^2 - 2v^2) \, dA \end{aligned}$$

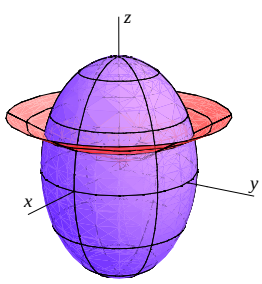
Since our region $\mathcal{R}$ is a disk, let's do this integral in polar coordinates. The disk $\mathcal{R}$ can be described as $0 \le r < 1$, $0 \le \theta < 2\pi$, so

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^1 (16 - 2r^2) \cdot r \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^1 (16r - 2r^3) \; dr \; d\theta \\
&= \int_0^{2\pi} \left(8r^2 - \frac{1}{2} r^4 \Big|_{r=0}^{r=1} \right) \; d\theta \\
&= \int_0^{2\pi} \frac{15}{2} \; d\theta \\
&= \boxed{15\pi}
\end{aligned}$$

5. *Let C be the curve of intersection of $2x^2 + 2y^2 + z^2 = 9$ with $z = \frac{1}{2}\sqrt{x^2 + y^2}$, oriented counterclockwise when viewed from above, and let $\vec{F}(x, y, z) = \langle 3y, 2yz, xz^3 + \sin z^2 \rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

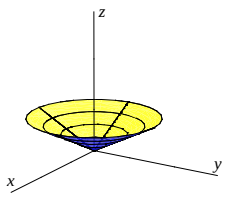
Solution. Again, we'll use Stokes' Theorem. The curl of the given vector field is $\operatorname{curl} \vec{F} = \left\langle -2y, -z^3, -3 \right\rangle$.

We'll start by thinking of an oriented surface $\mathcal{S}$ whose (oriented) boundary is the given curve C. The curve is the intersection of an ellipsoid with a cone:

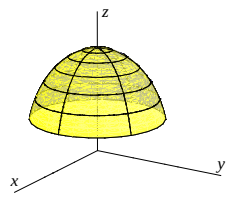


It appears from the picture that the curve lies in a plane parallel to the xy-plane. To verify this, let's look at the equations $2x^2 + 2y^2 + z^2 = 9$ and $z = \frac{1}{2}\sqrt{x^2 + y^2}$ defining the curve. The second equation can be rewritten as $x^2 + y^2 = 4z^2$. Plugging this into the first equation, $9z^2 = 9$, so $z = \pm 1$. Since the cone is only defined for $z \ge 0$, we know the intersection is where $z = 1$, in which case $x^2 + y^2 = 4$.

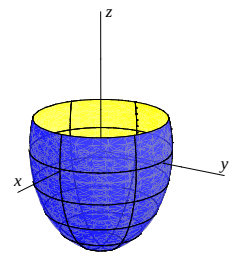
There are many surfaces whose boundary is the given curve C. From what we know so far, there are several surfaces that might come to mind:



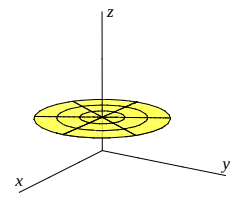
The part of the cone inside the ellipsoid.



The part of the ellipsoid lying above the curve.



The part of the ellipsoid lying below the curve.

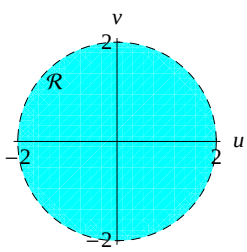


The part of the plane $z = 1$ lying inside $x^2 + y^2 = 4$.

Each of these should be oriented so that the yellow side is the "positive" side. The last is probably the simplest, so let's use that. To have the yellow side be the "positive" side, we want the normals to point upward. Thus, we have rewritten the original problem as:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the plane $z = 1$ lying inside $x^2 + y^2 = 4$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \left\langle -2y, -z^3, -3 \right\rangle$.

To do this, we first parameterize the surface. Since it is part of the plane $z = 1$, we can parameterize it by $\vec{r}(u, v) = \langle u, v, 1 \rangle$. Since we want only the part inside $x^2 + y^2 = 4$, the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is the disk $u^2 + v^2 < 4$:



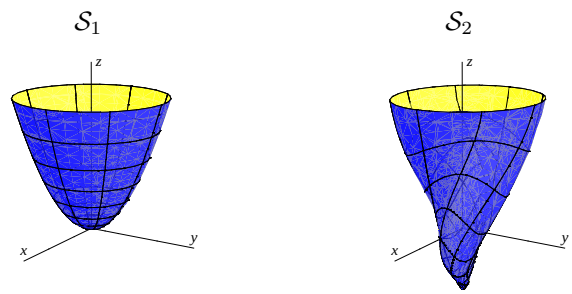
Next, we see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, 0 \rangle \\ \vec{r}_v &= \langle 0, 1, 0 \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 0, 0, 1 \rangle \end{aligned}$$

This always points upward, so our parameterization matches the orientation we want. Therefore, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \; dA \\ &= \iint_{\mathcal{R}} \langle -2v, -1, -3 \rangle \cdot \langle 0, 0, 1 \rangle \; dA \\ &= \iint_{\mathcal{R}} -3 \; dA \\ &= -3 \iint_{\mathcal{R}} 1 \; dA \\ &= -3(\text{area of } \mathcal{R}) \text{ by the worksheet "Double Integrals", \#2(a)} \\ &= \boxed{-12\pi} \end{aligned}$$

6. *The two surfaces shown have the same boundary. Suppose they are both oriented so that the light side is the "positive" side. Is the following reasoning correct? "Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same (oriented) boundary, the flux integrals $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$ must be equal for any vector field $\vec{G}$. Therefore, you can compute any flux integral using the simpler surface."*



Solution. $\boxed{\text{False.}}$ The statement *is* true if the vector field $\vec{G}$ is the curl of some other vector field, say $\vec{G} = \operatorname{curl} \vec{F}$. In that case, if C is the (properly oriented) boundary, Stokes' Theorem says that $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ are both equal to $\int_C \vec{F} \cdot d\vec{r}$. But there's no reason that $\vec{G}$ has to be the curl of some other vector field, so the statement is false in general.

Stokes' Theorem

Cast of Players:

S – an oriented, piecewise-smooth surface

C – a simple, closed, piecewise-smooth curve that bounds S

$\mathbf{F}$ – a vector field whose components have continuous derivatives in an open region of $\mathbf{R}^3$ containing S

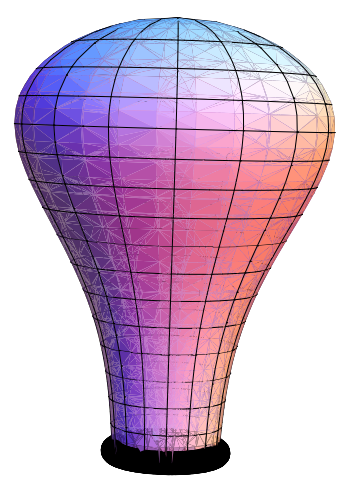
Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

1 Suppose C is the curve obtained by intersecting the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clockwise when viewed from above. Let S be the inside of this ellipse, oriented with the upward-pointing normal. If $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$, verify Stokes' theorem by computing both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2 Suppose S is that part of the plane $x + y + z = 1$ in the first octant, oriented with the upward-pointing normal, and let C be its boundary, oriented counter-clockwise when viewed from above. If $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$, verify Stokes' theorem by computing both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

3 Suppose S is a "light-bulb-shaped region" as follows. Imagine a light-bulb cut off at the base so that its boundary is the unit circle $x^2 + y^2 = 1$, oriented with the outward-pointing normal. (You can use either an old-fashioned light-bulb or a compact fluorescent if you're feeling green.) Suppose $\mathbf{F} = \left\langle e^{z^2 - 2z} x, \sin(xyz) + y + 1, e^{z^2} \sin(z^2) \right\rangle$. Compute the flux integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.



$\boxed{4}$ Suppose $\mathbf{F} = \langle -y, x, z \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 25$ below the plane $z = 4$, oriented with the outward-pointing normal (so that the normal at $(5, 0, 0)$ is $\mathbf{i}$). Compute the flux integral $\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.

$\boxed{5}$ Suppose S_1 and S_2 are two oriented surfaces that share C as boundary. What can you say about $\iint\limits_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ and $\iint\limits_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$?

$\boxed{6}$ Suppose S_1 and S_2 are two oriented surfaces that share C as boundary. Is it true that $\iint\limits_{S_1} \mathbf{G} \cdot d\mathbf{S} = \iint\limits_{S_2} \mathbf{G} \cdot d\mathbf{S}$ for any vector field $\mathbf{G}$? That is, can you *always* choose the easiest surface to work with when computing flux integrals over a surface with boundary?

$\boxed{7}$ Suppose S is a closed surface (that is, a surface without a boundary). Must $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = 0$?

Stokes' Theorem – Answers and Solutions

$\boxed{1}$ There are two integrals to compute here, so we do them both.

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ The ellipse is a graph (using $z = x$) over the unit circle in the xy-plane. Thus we can parameterize it as $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(t) \rangle$ for $0 \le t \le 2\pi$. Since $\mathbf{F} = \langle x, z, 2y \rangle$, we get

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle \cos(t), \cos(t), 2\sin(t) \rangle \\ d\mathbf{r} &= \langle -\sin(t), \cos(t), -\sin(t) \rangle \, dt \end{aligned}$$

and so

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \big(-\sin(t)\cos(t) + \cos^2(t) - 2\sin^2(t) \big) \, dt \\ &= \big(-\sin(t)\cos(t) + 1 - 3\sin^2(t) \big) \, dt. \end{aligned}$$

Thus

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \big(-\sin(t)\cos(t) + 1 - 3\sin^2(t) \big) \, dt \\ &= \left(-\frac{1}{2}\sin^2(t) + t - \frac{3}{2}t + \frac{3}{4}\sin(2t) \right) \Bigg|_0^{2\pi} \\ &= -\pi. \end{aligned}$$

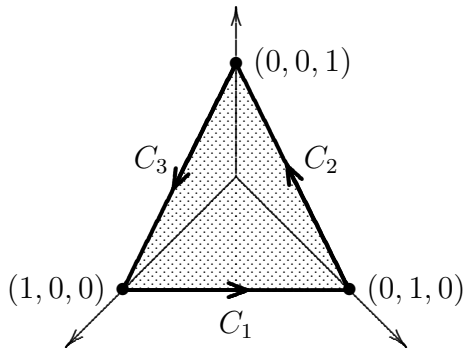
The flux integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ Again the elliptical disk is a graph (using $z = x$) over the unit disk in the xy-plane. Thus we can parameterize it as $\mathbf{r}(x, y) = \langle x, y, x \rangle$ for $x^2 + y^2 \le 1$. Since $\mathbf{F} = \langle x, z, 2y \rangle$, we get $\operatorname{curl} \mathbf{F} = \langle 1, 0, 0 \rangle$ and $\mathbf{r}_x \times \mathbf{r}_y = \langle -1, 0, 1 \rangle$. Thus

$$\begin{aligned} \iint_C \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\text{unit disk}} \langle 1, 0, 0 \rangle \cdot \langle -1, 0, 1 \rangle \, dx \, dy \\ &= -\iint_{\text{unit disk}} 1 \, dx \, dy \\ &= -\pi, \end{aligned}$$

since the last integral is simply the area $\pi(1)^2 = \pi$ of the unit disk.

Note that the two integrals agree. Another victory for Stokes' theorem!

$\boxed{2}$ There are two integrals to compute here, so we do them both. Here's a picture of the surface and curve, so we're all on the same page:



The region S is the dotted triangle (with the upward normal coming straight toward the viewer) and the curve C is the union $C_1 \cup C_2 \cup C_3$.

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ This integral will really be the sum of three separate integrals, over each of C_1, C_2, and C_3. We begin with C_1. A simple parameterization of this line segment is

$$\begin{aligned}\mathbf{r}(t) &= \text{starting point} + t\,(\text{ ending point } - \text{ starting point })\\ &= \langle 1,0,0\rangle + t\,(\langle 0,1,0\rangle - \langle 0,1,0\rangle)\\ &= \langle 1-t, t, 0\rangle.\end{aligned}$$

Thus $d\mathbf{r} = \langle -1,1,0\rangle\ dt$. In terms of this parameterization, the vector field $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2\rangle$ becomes

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= \left\langle (1-t)^2 - t^2, t^2 - 0^2, 0^2 - (1-t)^2\right\rangle\\ &= \left\langle 1-2t, t^2, -(1-t)^2\right\rangle\end{aligned}$$

Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \left\langle 1-2t, t^2, -(1-t)^2\right\rangle \cdot \langle -1,1,0\rangle\ dt\\ &= \left(t^2 + 2t - 1\right)\ dt.\end{aligned}$$

Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^2 + 2t - 1\right)\ dt = \frac{1}{3}.$$

The curve C_2 is similar: it's parameterized by $\mathbf{r}(t) = \langle 0, 1-t, t\rangle$, so

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \left\langle -(1-t)^2, 1-2t, t^2\right\rangle \cdot \langle 0,-1,1\rangle\ dt\\ &= \left(t^2 + 2t - 1\right)\ dt.\end{aligned}$$

Thus

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^2 + 2t - 1\right)\ dt = \frac{1}{3},$$

and very similarly $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}$ as well. Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

The flux integral $\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ This triangular surface is a graph (using $z = 1 - x - y$) over the triangle $T = \{(x, y) \ : \ 0 \le y \le 1 - x, \ 0 \le x \le 1\}$ in the first quadrant of the xy-plane. Thus we can parameterize it as $\mathbf{r}(x, y) = \langle x, y, 1 - x - y \rangle$ for $(x, y) \in T$. Since $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$, we get

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & y^2 - z^2 & z^2 - x^2 \end{vmatrix} = \langle 2z, 2x, 2y \rangle .$$

A similar computation shows that and $\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 1, 1 \rangle$ (and, since the $\mathbf{k}$ coefficient is positive, this is the upward-pointing normal). Thus

$$\begin{aligned} \iint_C \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_T \langle 2(1 - x - y), 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle \, dA \\ &= \int_0^1 \int_0^{1-x} 2 \, dy \, dx \\ &= 1, \end{aligned}$$

as before. More success for Stokes' theorem!

$\boxed{3}$ The point of this problem is to find use Stokes' theorem to avoid computing the flux integral over S (whatever confusing surface that could be) and instead compute the line integral over the unit circle C in the xy-plane. We use the parameterization $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$, so

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle \cos(t), \sin(t) + 1, 0 \rangle \\ d\mathbf{r} &= \langle -\sin(t), \cos(t), 0 \rangle \, dt \\ \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \cos(t) \, dt. \end{aligned}$$

Thus

$$\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \cos(t) \, dt = 0.$$

That was pretty easy.

$\boxed{4}$ Again we integrate the line integral over the boundary curve C rather than the flux integral over the (more complicated) surface S. The boundary curve is the circle $x^2 + y^2 + 4^2 = 25$ (or $x^2 + y^2 = 9$) in the plane $z = 4$, but a note of caution is in order. The natural parameterization

(or the one we usually think of) is $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4\rangle$ actually parameterizes $-C$ (that is, C with the opposite orientation)! Why is that? Imagine a person walking this boundary with their head in the normal (outward) direction. The remaining part of the sphere is on their *right* if they're walking counter-clockwise. It should be on their left, so they should be walking clockwise.

We'll calculate $\oint_{-C} \mathbf{F} \cdot d\mathbf{r}$ anyway, since we like the parameterization. In terms of this parameterization,

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle -3\sin(t), 3\cos(t), 4\rangle \\ d\mathbf{r}(t) &= \langle -3\sin(t), 3\cos(t), 0\rangle \; dt \\ \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}(t) &= 9 \; dt. \end{aligned}$$

Thus

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 9 \; dt = 18\pi,$$

and so $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -18\pi.$

$\boxed{5}$ If the boundaries of S_1 and S_2 are both C (with the same orientation!), then two applications of Stokes' theorem means that

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

so these two flux integrals must be the same.

$\boxed{6}$ What the previous problem was getting at was that this is always true when $\mathbf{G} = \operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$. But, alas, it is not true in general.

Here's a "simple" example: Let $\mathbf{G} = \langle 0, 0, z\rangle$ and S_1 be the unit square in the xy-plane:

$$S_1 = \{(x, y, z) \; : \; 0 \le x \le 1, \; 0 \le y \le 1, \; z = 0\}.$$

Then we'll let S_2 be the rest of the boundary of the unit cube, oriented so $S_1 \cup S_2$ encloses this cube and we have the outward-pointing normal. It should be clear that the flux across S_1 is zero (since $\mathbf{G} = \mathbf{0}$ on this surface), but a computation shows that, in fact, $\iint_{S_2} \mathbf{G} \cdot d\mathbf{S} = 1$.

We'll see in the next section that what is claimed here is true when $\operatorname{div} \mathbf{G} = 0$. But, like $\operatorname{curl} \mathbf{F} = \mathbf{0}$ implying that $\mathbf{F} = \nabla f$, it turns out that (under suitable assumptions) $\operatorname{div} \mathbf{G} = 0$ implies $\mathbf{G} = \operatorname{curl} \mathbf{F}$. (This will also give us an easy way to compute our "simple" example.)

$\boxed{7}$ No, $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ in this case, but not the given integral $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$.

If you're wondering how, say, the total net flow in or out of a closed surface can be something other than zero, then chances are you're too focused on water. A liquid like water is *incompressible*, so for water that flows into a region must be balanced by an equal amount that flows out (assuming that this region is totally submerged / full of water). But this is not true of all things that can flow; for example, electrical charge or temperature (two examples we had in homework due today) can have "sinks" or "sources" as we'll see Friday.

Preface

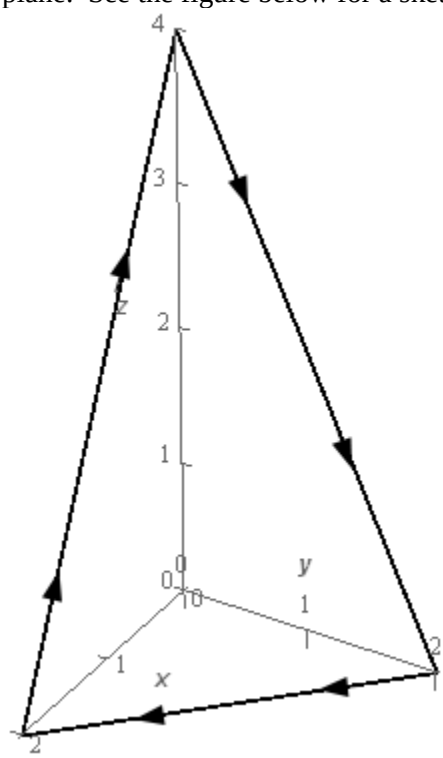
Here are a set of problems for my Calculus II notes. These problems do not have any solutions available on this site. These are intended mostly for instructors who might want a set of problems to assign for turning in. I try to put up both practice problems (with solutions available) and these problems at the same time so that both will be available to anyone who wishes to use them.

Stokes' Theorem

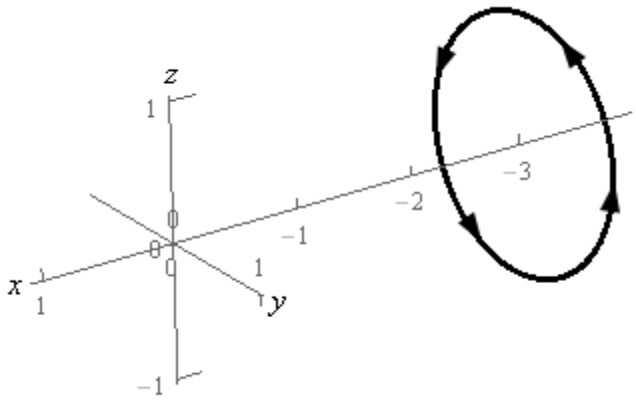
1. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$ where $\vec{F} = x^3\,\vec{i} + \left(4y - z^3y^3\right)\vec{j} + 2x\,\vec{k}$ and S is the portion of $z = x^2 + y^2 - 3$ below $z = 1$ with orientation in the negative z-axis direction.

2. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$ where $\vec{F} = 2y\,\vec{i} + 3x\,\vec{j} + \left(z - x\right)\vec{k}$ and S is the portion of $y = 11 - 3x^2 - 3z^2$ in front of $y = 5$ with orientation in the positive y-axis direction.

3. Use Stokes' Theorem to evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = \left(zx^3 - 2z\right)\vec{i} + xz\,\vec{j} + yx\,\vec{k}$ and C is is triangle with vertices $\left(0,0,3\right)$, $\left(0,2,0\right)$ and $\left(4,0,0\right)$. C has a clockwise rotation if you are above the triangle and looking down towards the xy-plane. See the figure below for a sketch of the curve.



4. Use Stokes' Theorem to evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = x^2\,\vec{i} - 4z\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 1 at $x = -3$ and perpendicular to the x-axis. C has a counter clockwise rotation if you are looking down the x-axis for the positive x-axis to the negative x-axis. See the figure below for a sketch of the curve.


z
1
−1
x
1
y
1
−1
−2
−3

Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my "class notes" they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

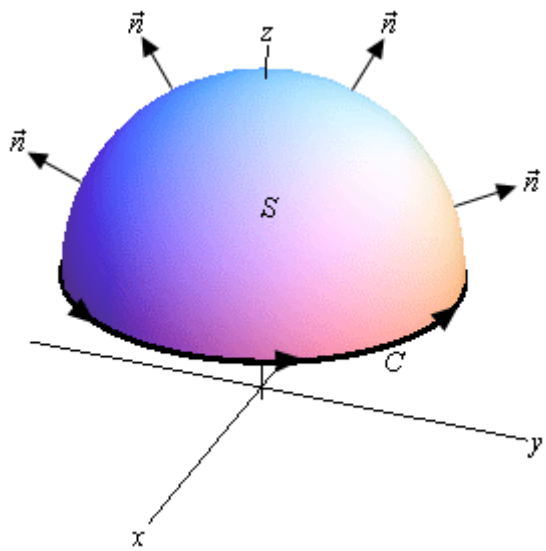
3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Stokes' Theorem

In this section we are going to take a look at a theorem that is a higher dimensional version of Green's Theorem. In Green's Theorem we related a line integral to a double integral over some region. In this section we are going to relate a line integral to a surface integral. However, before we give the theorem we first need to define the curve that we're going to use in the line integral.

Let's start off with the following surface with the indicated orientation.



Around the edge of this surface we have a curve C. This curve is called the **boundary curve**. The orientation of the surface S will induce the **positive orientation of C**. To get the positive orientation of C think of yourself as walking along the curve. While you are walking along the curve if your head is pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on C.

Now that we have this curve definition out of the way we can give Stokes' Theorem.

Stokes' Theorem

Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let $\vec{F}$ be a vector field then,

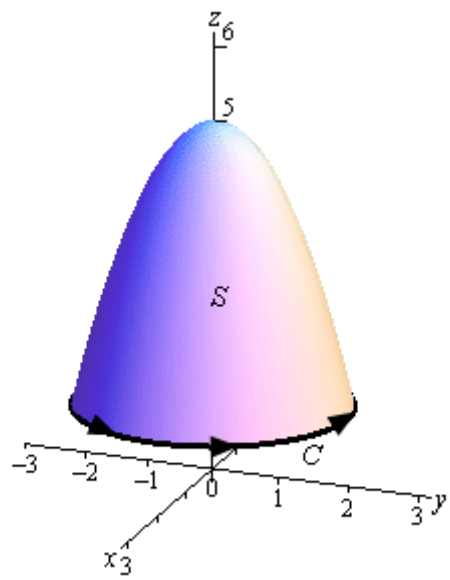
$$\int_C \vec{F} \bullet d\,\vec{r} = \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$$

In this theorem note that the surface S can actually be any surface so long as its boundary curve is given by C. This is something that can be used to our advantage to simplify the surface integral on occasion.

Let's take a look at a couple of examples.

Example 1 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$ where $\vec{F} = z^2\,\vec{i} - 3xy\,\vec{j} + x^3y^3\,\vec{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upwards.

Solution
Let's start this off with a sketch of the surface.



In this case the boundary curve C will be where the surface intersects the plane $z = 1$ and so will be the curve

$$1 = 5 - x^2 - y^2$$
$$x^2 + y^2 = 4 \qquad \text{at } z = 1$$

So, the boundary curve will be the circle of radius 2 that is in the plane $z = 1$. The parameterization of this curve is,

$$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + \vec{k}, \quad 0 \le t \le 2\pi$$

The first two components give the circle and the third component makes sure that it is in the plane $z = 1$.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S} = \int_C \vec{F} \bullet d\,\vec{r} = \int_0^{2\pi} \vec{F}\left(\vec{r}(t)\right) \bullet \vec{r}\,'(t)\,dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\vec{F}\left(\vec{r}(t)\right) = (1)^2\,\vec{i} - 3(2\cos t)(2\sin t)\,\vec{j} + (2\cos t)^3(2\sin t)^3\,\vec{k}$$
$$= \vec{i} - 12\cos t\sin t\,\vec{j} + 64\cos^3 t\sin^3 t\,\vec{k}$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\vec{r}'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$$
$$\vec{F}\left(\vec{r}(t)\right)\bullet\vec{r}'(t) = -2\sin t - 24\sin t\cos^2 t$$

We can now do the integral.

$$\iint_S \operatorname{curl}\vec{F}\bullet d\vec{S} = \int_0^{2\pi} -2\sin t - 24\sin t\cos^2 t\,dt$$
$$= \left(2\cos t + 8\cos^3 t\right)\Big|_0^{2\pi}$$
$$= 0$$

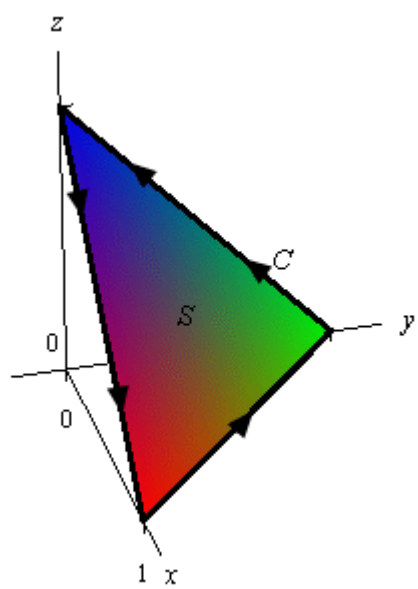
Example 2 Use Stokes' Theorem to evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F} = z^2\,\vec{i} + y^2\,\vec{j} + x\,\vec{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise rotation.

Solution

We are going to need the curl of the vector field eventually so let's get that out of the way first.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 2z\,\vec{j} - \vec{j} = (2z-1)\,\vec{j}$$

Now, all we have is the boundary curve for the surface that we'll need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.



Since the plane is oriented upwards this induces the positive direction on C as shown. The equation of this plane is,

$$x + y + z = 1 \qquad \Rightarrow \qquad z = g(x, y) = 1 - x - y$$

Now, let's use Stokes' Theorem and get the surface integral set up.

$$\begin{aligned}\int_C \vec{F} \bullet d\vec{r} &= \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S} \\ &= \iint_S (2z - 1)\, \vec{j} \bullet d\vec{S} \\ &= \iint_D (2z - 1)\, \vec{j} \bullet \frac{\nabla f}{\|\nabla f\|} \|\nabla f\| \, dA\end{aligned}$$

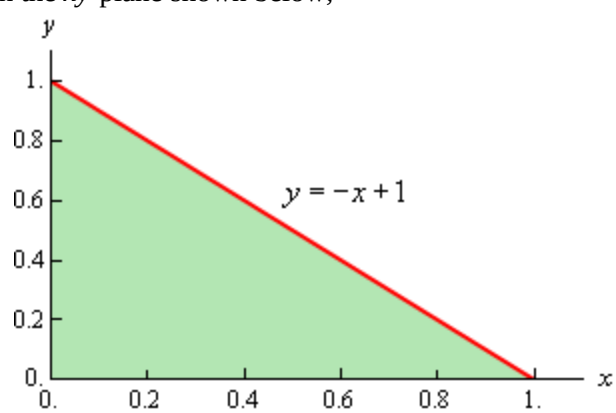
Okay, we now need to find a couple of quantities. First let's get the gradient. Recall that this comes from the function of the surface.

$$f(x, y, z) = z - g(x, y) = z - 1 + x + y$$

$$\nabla f = \vec{i} + \vec{j} + \vec{k}$$

Note as well that this also points upwards and so we have the correct direction.

Now, D is the region in the xy-plane shown below,



We get the equation of the line by plugging in $z = 0$ into the equation of the plane. So based on this the ranges that define D are,

$$0 \le x \le 1 \qquad 0 \le y \le -x + 1$$

The integral is then,

$$\begin{aligned}\int_C \vec{F} \bullet d\vec{r} &= \iint_D (2z - 1)\, \vec{j} \bullet \left(\vec{i} + \vec{j} + \vec{k}\right) dA \\ &= \int_0^1 \int_0^{-x+1} 2(1 - x - y) - 1 \, dy \, dx\end{aligned}$$

Don't forget to plug in for z since we are doing the surface integral on the plane. Finishing this out gives,

$$\begin{aligned}\int_C \vec{F} \bullet d\,\vec{r} &= \int_0^1 \int_0^{-x+1} 1 - 2x - 2y\,dy\,dx \\ &= \int_0^1 \left(y - 2xy - y^2 \right)\Big|_0^{-x+1} dx \\ &= \int_0^1 x^2 - x\,dx \\ &= \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 \right)\Bigg|_0^1 \\ &= -\frac{1}{6}\end{aligned}$$

In both of these examples we were able to take an integral that would have been somewhat unpleasant to deal with and by the use of Stokes' Theorem we were able to convert it into an integral that wasn't too bad.

Preface

Here are a set of practice problems for my Calculus III notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a number of places on the site.

1. If you'd like a pdf document containing the solutions go to the note page for the section you'd like solutions for and select the download solutions link from there. Or,

2. Go to the download page for the site http://tutorial.math.lamar.edu/download.aspx and select the section you'd like solutions for and a link will be provided there.

3. If you'd like to view the solutions on the web or solutions to an individual problem you can go to the problem set web page, select the problem you want the solution for. At this point I do not provide pdf versions of individual solutions, but for a particular problem you can select "Printable View" from the "Solution Pane Options" to get a printable version.

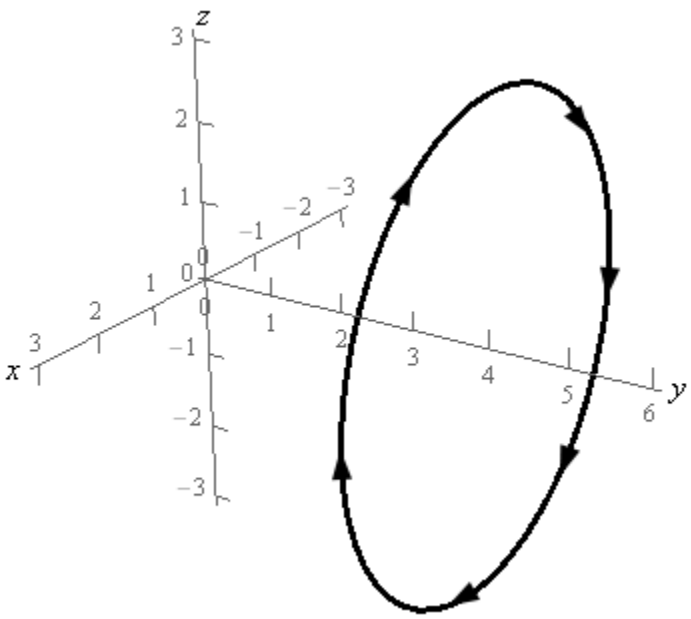
Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Stokes' Theorem

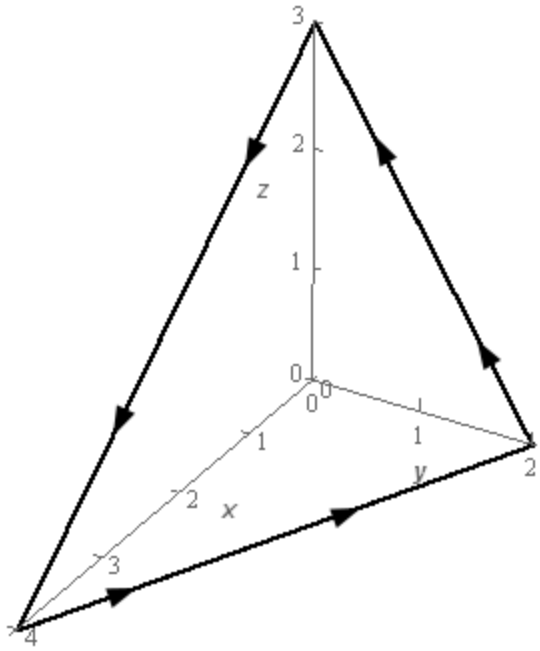
1. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$ where $\vec{F} = y\,\vec{i} - x\,\vec{j} + yx^3\,\vec{k}$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.

2. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$ where $\vec{F} = \left(z^2 - 1\right)\vec{i} + \left(z + xy^3\right)\vec{j} + 6\vec{k}$ and S is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative x-axis direction.

3. Use Stokes' Theorem to evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + \left(4y + 1\right)\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y-axis. C has a clockwise rotation if you are looking down the y-axis from the positive y-axis to the negative y-axis. See the figure below for a sketch of the curve.



4. Use Stokes' Theorem to evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = \left(3yx^2 + z^3\right)\vec{i} + y^2\,\vec{j} + 4yx^2\,\vec{k}$ and C is is triangle with vertices $\left(0,0,3\right)$, $\left(0,2,0\right)$ and $\left(4,0,0\right)$. C has a counter clockwise rotation if you are above the triangle and looking down towards the xy-plane. See the figure below for a sketch of the curve.


3
2
z
1
0
0
0
1
1
y
2
2
x
3
4

Preface

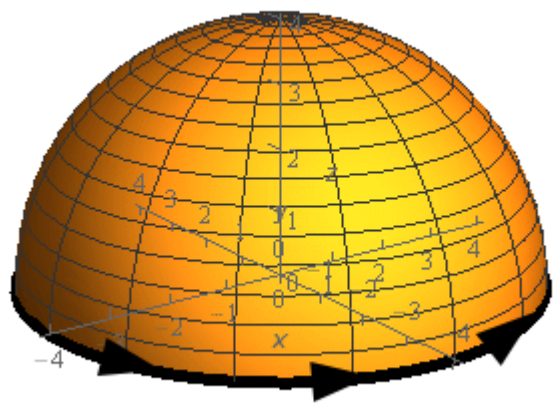
Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

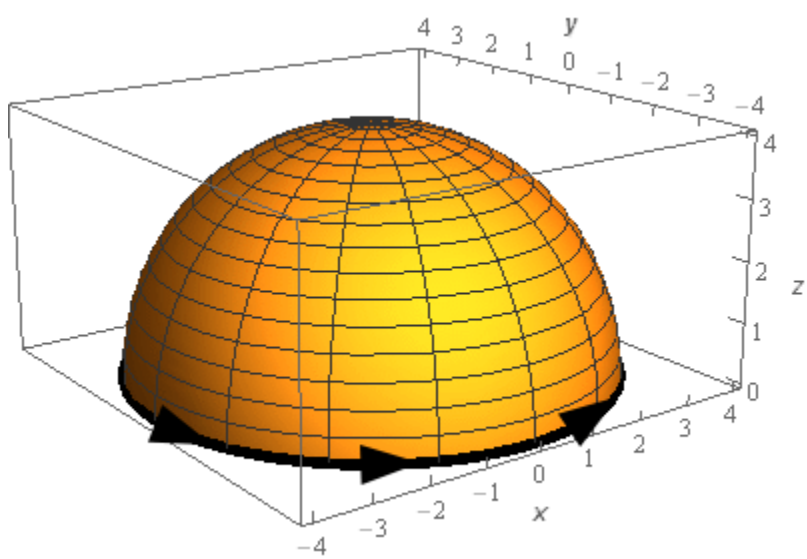
This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Stokes' Theorem

1. Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\vec{F} \bullet d\vec{S}$ where $\vec{F} = y\,\vec{i} - x\,\vec{j} + yx^3\,\vec{k}$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Because the orientation of the surface is upwards then all the normal vectors will be pointing outwards. So, if we walk along the edge of the surface, *i.e.* the curve C, in the direction indicated with our head

pointed away from the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore the direction indicated in the sketch is the positive orientation of *C*.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it upside down on a piece of paper on a table. Sketch a set of axis on the piece of paper that will represent the plane the cup/bowl is sitting on to really help with the visualization. Then cut out a little stick figure and put a face on the "front" side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with it's head pointing in the direction of the normal vectors (out away from the sphere/cup/bowl in our case) with its left hand over the surface. The direction that the "face" on the stick figure is facing is the direction you'd need to walk along the surface to get the positive orientation for *C*.

Step 2

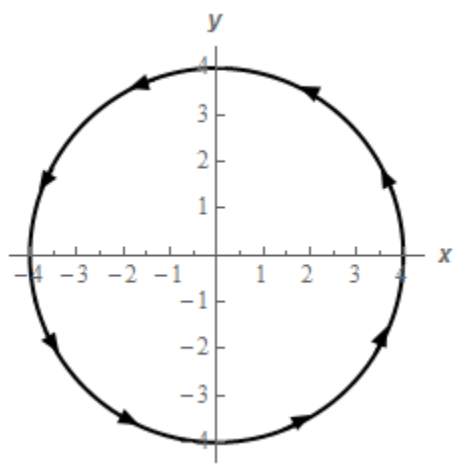
We are going to use Stokes' Theorem in the following direction.

$$\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S} = \int_C \vec{F} \bullet d\vec{r}$$

We've been given the vector field in the problem statement so we don't need to worry about that. We will need to deal with *C*.

Because *C* is just the curve along the bottom of the upper half of the sphere we can see that *C* in fact will be the intersection of the sphere and the *xy*-plane (*i.e.* $z = 0$). Therefore, *C* is just the circle of radius 4.

If we look at the sphere from above we get the following sketch of *C*.



The parameterization of *C* is given by,

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, 0 \rangle \qquad 0 \le t \le 2\pi$$

The *z* component of the parameterization is zero because *C* lies in the *xy*-plane.

Step 3

Since we know we'll need to eventually do the line integral we know we'll need the following dot product.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\,'\left(t\right) &= \left\langle 4\sin t, -4\cos t, 256\sin t\cos^3 t\right\rangle\bullet\left\langle -4\sin t, 4\cos t, 0\right\rangle \\ &= -16\sin^2 t - 16\cos^2 t \\ &= -16\end{aligned}$$

Don't forget to plug the parameterization of C into the vector field!

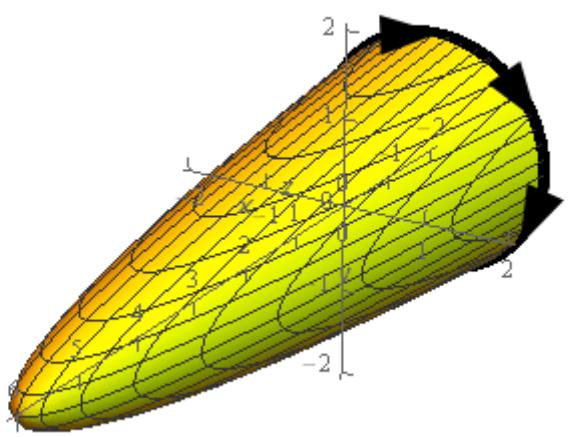
Step 4
Okay, let's go ahead and evaluate the integral using Stokes' Theorem.

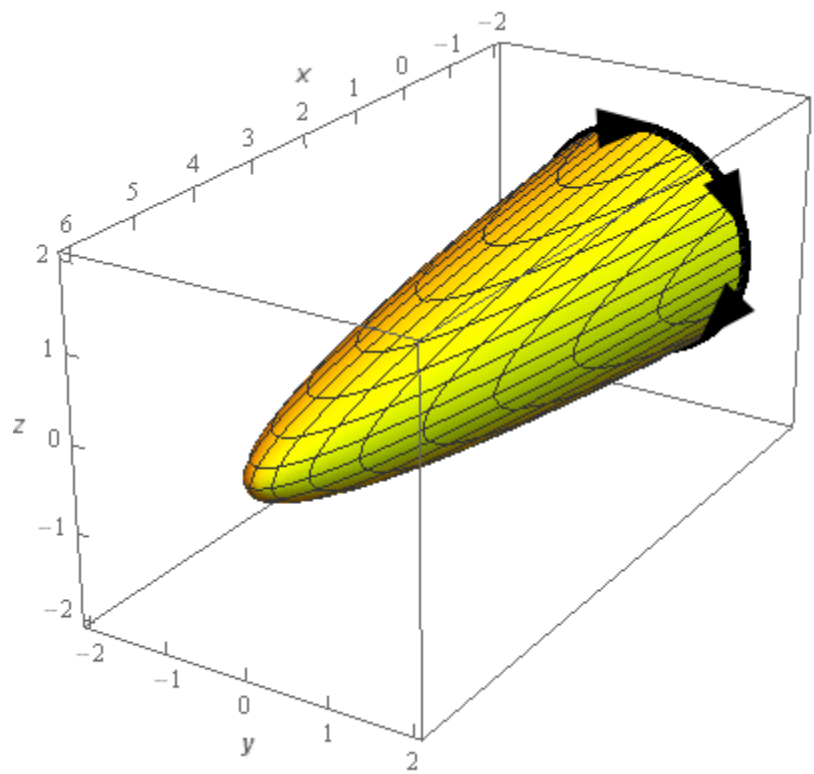
$$\begin{aligned}\iint_S \operatorname{curl}\vec{F}\bullet d\vec{S} &= \int_C \vec{F}\bullet d\vec{r} \\ &= \int_0^{2\pi} -16\,dt = \boxed{-32\pi}\end{aligned}$$

---

2. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\vec{F}\bullet d\vec{S}$ where $\vec{F} = \left(z^2 - 1\right)\vec{i} + \left(z + xy^3\right)\vec{j} + 6\vec{k}$ and S is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative x-axis direction.

Step 1
Let's start off with a quick sketch of the surface we are working with in this problem.





We included a sketch with traditional axes and a sketch with a set of "box" axes to help visualize the surface.

Because the orientation of the surface is towards the negative *x*-axis all the normal vectors will be pointing into the region enclosed by the surface. So, if we walk along the edge of the surface, *i.e.* the curve *C*, in the direction indicated with our head pointed into the region enclosed by the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore the direction indicated in the sketch is the positive orientation of *C*.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it on its side with a piece of paper behind it. Sketch a set of axis on the piece of paper that will represent the plane the cup/bowl is sitting in front of to really help with the visualization. Then cut out a little stick figure and put a face on the "front" side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with it's head pointing in the direction of the normal vectors (into the paraboloid/cup/bowl in our case) with its left hand over the surface. The direction that the "face" on the stick figure is facing is the direction you'd need to walk along the surface to get the positive orientation for *C*.

Step 2

We are going to use Stokes' Theorem in the following direction.

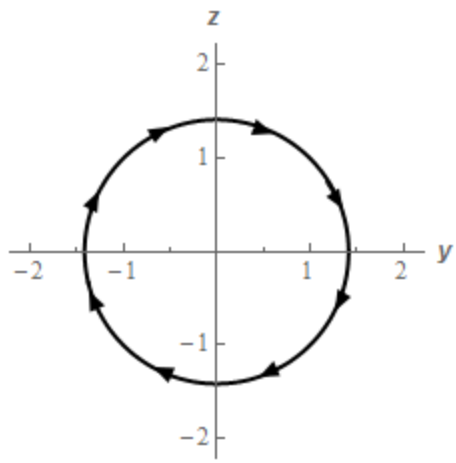
$$\iint_S \operatorname{curl} \vec{F} \bullet d\vec{S} = \int_C \vec{F} \bullet d\vec{r}$$

We've been given the vector field in the problem statement so we don't need to worry about that. We will need to deal with *C*.

In this case *C* is the curve we get by setting the two equations in the problem statement equal. Doing this gives,

$$-2 = 6 - 4y^2 - 4z^2 \quad \rightarrow \quad 4y^2 + 4z^2 = 8 \quad \Rightarrow \quad y^2 + z^2 = 2$$

We will see following sketch of *C* if we are in front of the paraboloid and look directly along the *x*-axis.



One possible parameterization of *C* is given by,

$$\vec{r}(t) = \left\langle -2, \sqrt{2}\sin t, \sqrt{2}\cos t \right\rangle \qquad 0 \le t \le 2\pi$$

The *x* component of the parameterization is –2 because *C* lies at $x = -2$.

Step 3
Since we know we'll need to eventually do the line integral we know we'll need the following dot product.

$$\begin{aligned}
\vec{F}\left(\vec{r}(t)\right)\bullet\vec{r}\,'(t) &= \left\langle 2\cos^2 t - 1, \sqrt{2}\cos t + 4\sqrt{2}\sin^3 t, 6 \right\rangle \bullet \left\langle 0, \sqrt{2}\cos t, -\sqrt{2}\sin t \right\rangle \\
&= \sqrt{2}\cos t\left(\sqrt{2}\cos t + 4\sqrt{2}\sin^3 t\right) - 6\sqrt{2}\sin t \\
&= 2\cos^2 t + 8\cos t\sin^3 t - 6\sqrt{2}\sin t \\
&= \left(1 - \sin(2t)\right) + 8\cos t\sin^3 t - 6\sqrt{2}\sin t
\end{aligned}$$

Don't forget to plug the parameterization of *C* into the vector field!

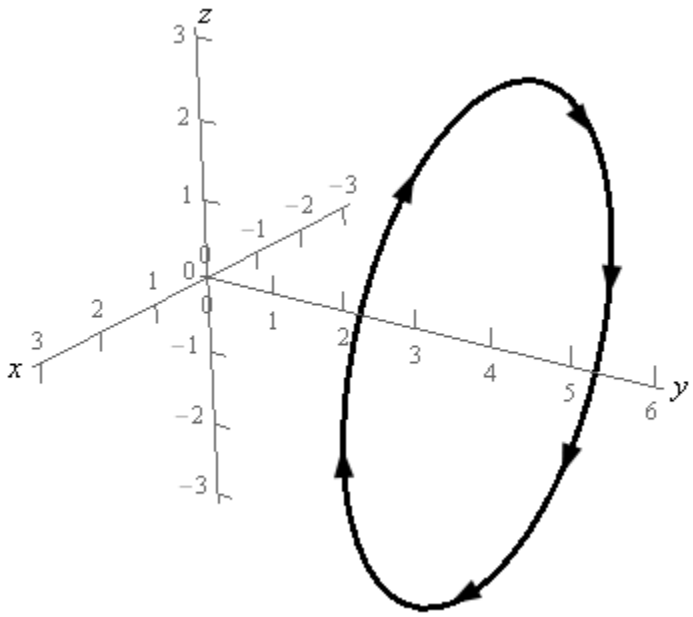
We also did a little simplification on the first term with an eye towards the integration.

Step 4
Okay, let's go ahead and evaluate the integral using Stokes' Theorem.

$$\iint_S \operatorname{curl}\vec{F}\bullet d\vec{S} = \int_C \vec{F}\bullet d\vec{r}$$
$$= \int_0^{2\pi} \left(1-\sin\left(2t\right)\right) + 8\cos t\sin^3 t - 6\sqrt{2}\sin t\,dt$$
$$= \left(t + \tfrac{1}{2}\cos\left(2t\right) + 2\sin^4 t + 6\sqrt{2}\sin t\,dt\right)\Big|_0^{2\pi} = \boxed{2\pi}$$

---

3. Use Stokes' Theorem to evaluate $\int_C \vec{F}\bullet d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + \left(4y+1\right)\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y-axis. C has a clockwise rotation if you are looking down the y-axis from the positive y-axis to the negative y-axis. See the figure below for a sketch of the curve.



Step 1
Okay, we are going to use Stokes' Theorem in the following direction.

$$\int_C \vec{F}\bullet d\vec{r} = \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S}$$

So, let's first compute $\operatorname{curl}\vec{F}$ since that is easy enough to compute and might be useful to have when we go to determine the surface S we're going to integrate over.

The curl of the vector field is then,

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & 4y+1 & xy \end{vmatrix} = x\vec{i} - y\vec{j} + z\vec{k} - y\vec{j} = \underline{x\vec{i} - 2y\vec{j} + z\vec{k}}$$

Step 2
Now we need to find a surface S with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation. This can seem to be a daunting task at times but it's not as bad as it might appear to be. First we know that the boundary curve needs to be a circle. This means that we're going to be looking for a surface whose cross section is a circle and we know of several surfaces that meet this requirement. We know that spheres, cones and elliptic paraboloids all have circles as cross sections.

The question becomes which of these surfaces would be best for us in this problem. To make this decision remember that we'll eventually need to plug this surface into the vector field and then take the dot product of this with normal vector (which will also come from the surface of course).

In general it won't be immediately clear from the curl of the vector field by itself which surface we should use and that is the case here. The curl of the vector field has all three components and none of them are that difficult to deal with but there isn't anything that suggests one surface might be easier than the other.

So, let's consider a sphere first. The issue with spheres is that its parameterization and normal vector are lengthy and many lead to messy integrands. So, because the curl of the vector field does have all three components to it which may well lead to long and/or messy integrands we'll not work with a sphere for this problem.

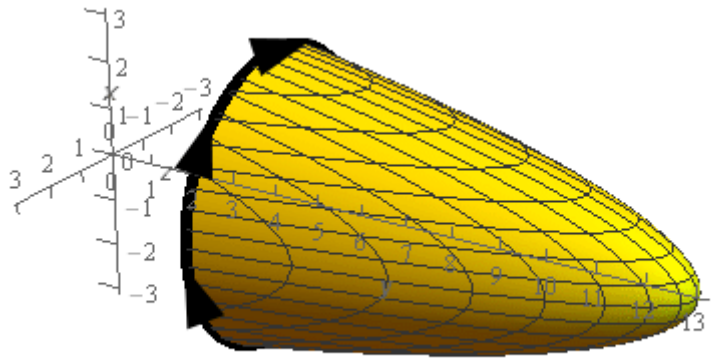
Now let's think about a cone. Equations of cones aren't that bad but they will involve a square root and in this case would need to be in the form $y = \sqrt{ax^2 + bz^2}$ because the boundary curve is centered on the y-axis. The normal vector will also contain roots and this will often lead to messy integrands. So, let's not work with a cone either in this problem.

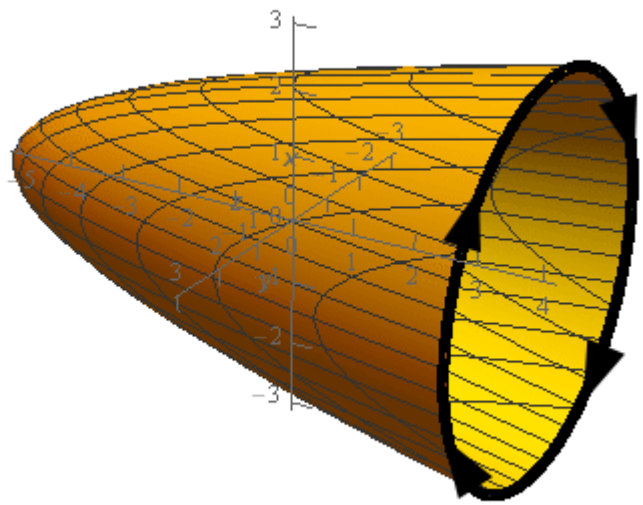
That leaves elliptic paraboloids and we probably should have considered them first. The equations are simple and the normal vectors are even simpler so they seem like a good choice of surface for this problem.

Note that we're not saying that spheres and cones are never good choices for the surface. For some vector fields the curl may end up being very simple with one of these surfaces and so they would be perfectly good choices.

Step 3
We have two possibilities for elliptic paraboloids that we could use here. Both will be centered on the y-axis but one will open in the negative y direction while the other will open in the positive y direction.

Here is a couple of sketches of possible elliptic paraboloids we could use here.





Let's get an equation for each of these. Note that for each of these if we set the equation of the paraboloid and the plane $y = 4$ equal we need to get the circle $x^2 + z^2 = 9$ since this is the boundary curve that should occur at $y = 4$.

Let's get the equation of the first paraboloid (the one that opens in the negative *y* direction. We know that the equation of this paraboloid should be $y = a - x^2 - z^2$ for some value of *a*. As noted if we set this equal to $y = 4$ and do some simplification we know what equation we should get. So, let's set the two equations equal.

$$4 = a - x^2 - z^2 \qquad \rightarrow \qquad x^2 + z^2 = a - 4 = 9 \qquad \rightarrow \qquad a = 13$$

As shown we know that the $a - 4$ should be 9 and so we must have $a = 13$. Therefore the equation of the paraboloid that open in the negative *y* direction is,

$$y = 13 - x^2 - z^2$$

Next, let's get the equation of the paraboloid that opens in the positive *y* direction. The equation of this paraboloid will be in the form $y = x^2 + y^2 + a$ for some *a*. Setting this equal to $y = 4$ gives,

$$4 = x^2 + z^2 + a \qquad \rightarrow \qquad x^2 + z^2 = 4 - a = 9 \qquad \rightarrow \qquad a = -5$$

The equation of the paraboloid that opens in the positive *y* direction is then,

$$y = x^2 + z^2 - 5$$

Either of these surfaces could be used to do this problem.

Step 4
We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, *C*, that matches the orientation that was given in the problem statement.

We'll find the normal vectors for each surface despite the fact that we really only need to do it for one of them since we only need one of the surfaces to do the problem as noted in the previous step. Determining the orientation of the surface can be a little tricky for some folks so doing an extra one might help see what's going on here.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the surface itself. We now need to determine if we are walking along the outside of the surface or the inside of the surface.

If we are walking along the outside of the surface then our heads, and hence the normal vectors, will be pointing away from the region enclosed by the surface. On the other hand if we are walking along the inside of the surface then our heads, and hence the normal vectors, will be pointing into the region enclosed by the surface.

To help visualize this for our two surfaces it might help to get a cup or bowl that we can use to represent the surface. The edge of the cup/bowl will then represent the boundary curve. Next cut out a stick figure and put a face on one side so we know which direction we'll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, put the cup/bowl on its side so it looks vaguely like the surface we're working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Do we need to put the stick figure on the inside or outside of the cup/bowl to do this?

Okay, let's do this for the first surface, $y = 13 - x^2 - z^2$. In this case our stick figure would need to be standing on the inside of the cup/bowl/surface. Therefore the normal vectors on the surface would all need to be point in towards the region enclosed by the surface. This also will mean that all the normal vectors will need to have a negative *y* component. Again, to visualize this take the stick figure and move it into the region and toward the end of cup/bowl/surface and you'll see it start to point more and more in the negative *y* direction (and hence will have a negative *y* component). Note that the *x* and *z* component can be either positive or negative depending on just where we are on the interior of the surface.

Now, let's take a look at the first surface, $y = x^2 + z^2 - 5$. For this surface our stick figure would need to be standing on the outside of the cup/bowl/surface. So, in this case, the normal vectors would point out away from the region enclosed by the surface. These will also have a negative *y* component and you can use the method we discussed in the above paragraph to help visualize this.

Step 5
We now need to start thinking about actually computing the integral. We'll write the equation of the surface as,

$$f\left(x, y, z\right) = 13 - x^2 - z^2 - y = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle -2x, -1, -2z \right\rangle}{\left\|\nabla f\right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the *y* component is negative.

Next, we'll need to compute the following dot product.

$$\begin{aligned}
\operatorname{curl}\vec{F}\bullet\vec{n} &= \left\langle x, -2\left(13 - x^2 - z^2\right), z\right\rangle \bullet \frac{\left\langle -2x, -1, -2z \right\rangle}{\left\|\nabla f\right\|} \\
&= \tfrac{1}{\left\|\nabla f\right\|}\left(-2x^2 + 2\left(13 - x^2 - z^2\right) - 2z^2\right) \\
&= \tfrac{1}{\left\|\nabla f\right\|}\left(26 - 4x^2 - 4z^2\right)
\end{aligned}$$

Step 6
Now, applying Stokes' Theorem to the integral and converting to a "normal" double integral gives,

$$\begin{aligned}
\int_C \vec{F}\bullet d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S} \\
&= \iint_S \tfrac{1}{\left\|\nabla f\right\|}\left(26 - 4x^2 - 4z^2\right) dS \\
&= \iint_D \tfrac{1}{\left\|\nabla f\right\|}\left(26 - 4x^2 - 4z^2\right)\left\|\nabla f\right\| dA \\
&= \iint_D 26 - 4x^2 - 4z^2 dA
\end{aligned}$$

Step 7
To finish this integral out then we'll need to convert to polar coordinates using the following polar coordinates.

$$x = r\cos\theta \qquad z = r\sin\theta \qquad x^2 + z^2 = r^2$$

In this case D is just the disk $x^2 + z^2 \le 9$ and so the limits for the integral are,

$$\begin{aligned} 0 &\le \theta \le 2\pi \\ 0 &\le r \le 3 \end{aligned}$$

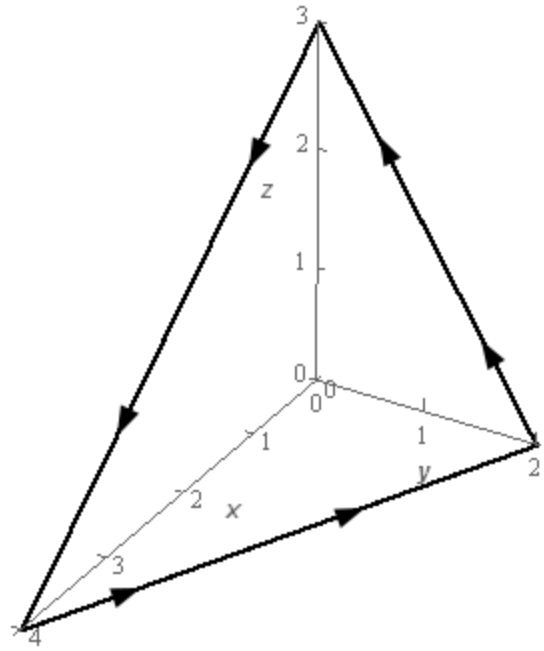
The integral is then,

$$\begin{aligned} \int_C \vec{F} \bullet d\vec{r} &= \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S} \\ &= \int_0^{2\pi} \int_0^3 \left(26 - 4r^2\right) r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^3 26r - 4r^3\, dr\, d\theta \\ &= \int_0^{2\pi} \left(13r^2 - r^4\right)\Big|_0^3 d\theta \\ &= \int_0^{2\pi} 36\, d\theta \\ &= \boxed{72\pi} \end{aligned}$$

Don't forget to pick up an extra r from converting the dA to polar coordinates.

---

4. Use Stokes' Theorem to evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = \left(3yx^2 + z^3\right)\vec{i} + y^2\,\vec{j} + 4yx^2\,\vec{k}$ and C is is triangle with vertices $(0,0,3)$, $(0,2,0)$ and $(4,0,0)$. C has a counter clockwise rotation if you are above the triangle and looking down towards the xy-plane. See the figure below for a sketch of the curve.



Step 1
Okay, we are going to use Stokes' Theorem in the following direction.

$$\int_C \vec{F} \bullet d\vec{r} = \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S}$$

So, let's first compute $\operatorname{curl} \vec{F}$ since that is easy enough to compute and might be useful to have when we go to determine the surface *S* we're going to integrate over.

The curl of the vector field is then,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3yx^2 + z^3 & y^2 & 4yx^2 \end{vmatrix} = 4x^2\vec{i} + 3z^2\vec{j} - 3x^2\vec{k} - 8yx\vec{j} = \underline{4x^2\vec{i} + \left(3z^2 - 8yx\right)\vec{j} - 3x^2\vec{k}}$$

Step 2
Now we need to find a surface *S* with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation.

In this case we can see that the triangle looks like the portion of a plane and so it makes sense that we use the equation of the plane containing the three vertices for the surface here.

The curl of the vector field looks a little messy so using a plane here might be the best bet from this perspective as well. It will (hopefully) not make the curl of the vector field any messier and the normal vector, which we'll get from the equation of the plane, will be simple and so shouldn't make the curl of the vector field any worse.

Step 3
Determining the equation of the plane is pretty simple. We have three points on the plane, the vertices, and so we can quickly determine the equation.

First, let's "label" the points as follows,

$$P = (4,0,0) \qquad Q = (0,2,0) \qquad R = (0,0,3)$$

Then two vectors that must lie in the plane are,

$$\overrightarrow{QP} = \langle 4,-2,0 \rangle \qquad \overrightarrow{QR} = \langle 0,-2,3 \rangle$$

To write the equation of a plane recall that we need a normal vector to the plane. Now, we know that the cross product of these two vectors will be orthogonal to both of the vectors. Also, since both of the vectors lie in the plane the cross product will also be orthogonal, or normal, to the plane. In other words, we can use the cross product of these two vectors as the normal vector to the plane.

The cross product is,

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = -6\vec{i} - 12\vec{j} - 8\vec{k}$$

Now we can use any of the points in our equation. We'll use Q for our point. The equation of the plane is then,

$$-6(x-0) - 12(y-2) - 8(z-0) = 0 \qquad \rightarrow \qquad \begin{aligned} -6x - 12y - 8z &= -24 \\ \underline{3x + 6y + 4z = 12} \end{aligned}$$

Note that we divided the equation by –2 to make the equation a little "nicer" to work with.

Step 4
We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, C, that matches the orientation that was given in the problem statement.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the plane. We now need to determine if we are walking along the top or bottom of the plane.

If we are walking along the top of the plane then our heads, and hence the normal vectors, will be pointing in a generally upwards direction. On the other hand if we are walking along the bottom of the plane then our heads, and hence the normal vectors, will be pointing generally downwards.

To help visualize this for our plane it might help to cut out a triangular piece of paper that we can use to represent the plane. The edge of the piece of paper will then represent the boundary curve. Next cut out a

stick figure and put a face on one side so we know which direction we'll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, hold the piece of paper so that it looks vaguely like the surface we're working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Doing this we'll quickly see that we must be walking along the top of the surface. Therefore the normal vectors on the surface need to be pointing in a generally upwards direction (and hence will have a positive *z* component).

Step 5
We now need to start thinking about actually computing the integral. We'll write the equation of the surface as,

$$z = 3 - \tfrac{3}{4}x - \tfrac{3}{2}y$$

Recall that if we aren't going to parameterize the surface we need it to be written as $z = g\left(x, y\right)$ so that the magnitude of the normal vector will eventually cancel.

Now, that we have the surface written in the "proper" form let's define,

$$f\left(x, y, z\right) = z - 3 + \tfrac{3}{4}x + \tfrac{3}{2}y = 0$$

The unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\left\|\nabla f\right\|} = \frac{\left\langle \frac{3}{4}, \frac{3}{2}, 1\right\rangle}{\left\|\nabla f\right\|}$$

We didn't compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the *z* component is positive.

Next, we'll need to compute the following dot product.

$$\begin{aligned}
\operatorname{curl}\vec{F}\bullet\vec{n} &= \left\langle 4x^2, \left(3\left(3 - \tfrac{3}{4}x - \tfrac{3}{2}y\right)^2 - 8yx\right), -3x^2\right\rangle \bullet \frac{\left\langle \frac{3}{4}, \frac{3}{2}, 1\right\rangle}{\left\|\nabla f\right\|} \\
&= \tfrac{1}{\left\|\nabla f\right\|}\left(3x^2 + \tfrac{9}{2}\left(3 - \tfrac{3}{4}x - \tfrac{3}{2}y\right)^2 - 12xy - 3x^2\right) \\
&= \tfrac{1}{\left\|\nabla f\right\|}\left[\tfrac{9}{2}\left(3 - \tfrac{3}{4}x - \tfrac{3}{2}y\right)^2 - 12xy\right]
\end{aligned}$$

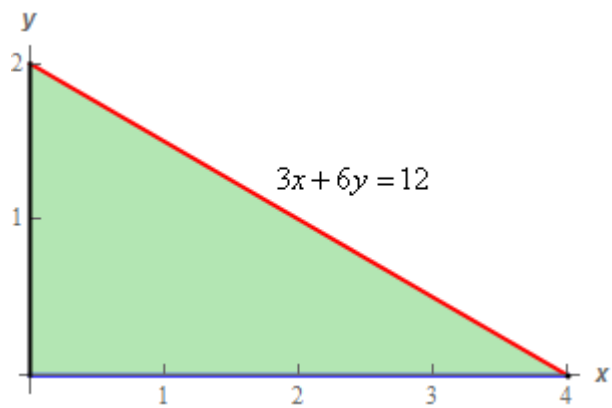
Don't forget to plug the equation of the surface into the curl of the vector field.

Step 6
Now, applying Stokes' Theorem to the integral and converting to a "normal" double integral gives,

$$\begin{aligned}\int_C \vec{F}\bullet d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S}\\ &= \iint_S \tfrac{1}{\|\nabla f\|}\left[\tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\right]dS\\ &= \iint_D \tfrac{1}{\|\nabla f\|}\left[\tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\right]\|\nabla f\|\,dA\\ &= \iint_D \tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2-12xy\,dA\end{aligned}$$

Step 7

To finish this integral we just need to determine *D*. In this case *D* just the triangle in the *xy*-plane that lies below the plane. Here is a quick sketch of *D*.



The integral doesn't seem to suggest one integration order over the other so let's use the following set of limits for our integral.

$$\begin{aligned}&0\le x\le 4\\ &0\le y\le 2-\tfrac{1}{2}x\end{aligned}$$

The integral is then,

$$\begin{aligned}
\int_C \vec{F}\bullet d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\bullet d\vec{S} \\
&= \int_0^4 \int_0^{2-\frac{1}{2}x} \tfrac{9}{2}\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^2 - 12xy\,dy\,dx \\
&= \int_0^4 \left(-\left(3-\tfrac{3}{4}x-\tfrac{3}{2}y\right)^3 - 6xy^2\right)\Big|_0^{2-\frac{1}{2}x} dx \\
&= \int_0^4 \left(3-\tfrac{3}{4}x\right)^3 - 6x\left(2-\tfrac{1}{2}x\right)^2 dx \\
&= \int_0^4 \left(3-\tfrac{3}{4}x\right)^3 - 24x + 12x^2 - \tfrac{3}{2}x^3\,dx \\
&= \left(-\tfrac{1}{3}\left(3-\tfrac{3}{4}x\right)^4 - 12x^2 + 4x^3 - \tfrac{3}{8}x^4\right)\Big|_0^4 \\
&= \boxed{-5}
\end{aligned}$$

Stokes' Theorem Examples

Stokes' Theorem relates surface integrals and line integrals.

STOKES' THEOREM

Let $\mathbf{F}$ be a vector field. Let S be an oriented surface, and let C be the boundary curve of S, oriented using the right-hand rule. Then:

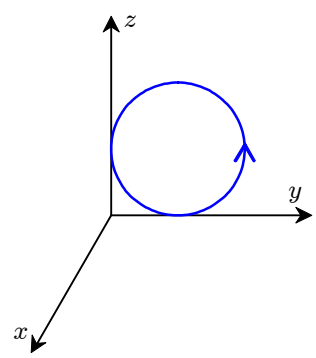
$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{A} \;=\; \oint_C \mathbf{F} \cdot d\mathbf{s}$$

EXAMPLE 1 Let C be the curve defined by the parametric equations

$$\begin{aligned} x &= 0 \\ y &= 2 + 2\cos t \qquad 0 \le t \le 2\pi \\ z &= 2 + 2\sin t \end{aligned}$$

Use Stokes' Theorem to evaluate $\displaystyle\int_C x^2 e^{5z}\, dx + x\cos y\, dy + 3y\, dz$.

SOLUTION The parametric equations above describe a circle of radius 2 on the yz-plane:



Let D be the disc whose boundary is the given circle. By Stokes' Theorem:

$$\int_C \left(x^2 e^{5z}\,\mathbf{i} + x\cos y\,\mathbf{j} + 3y\,\mathbf{k}\right) \cdot d\mathbf{s} \;=\; \iint_D \operatorname{curl}\left(x^2 e^{5z}\,\mathbf{i} + x\cos y\,\mathbf{j} + 3y\,\mathbf{k}\right) \cdot d\mathbf{A}$$

We compute the curl:

$$\operatorname{curl}\left(x^2 e^{5z}\,\mathbf{i} + x\cos y\,\mathbf{j} + 3y\,\mathbf{k}\right) \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 e^{5z} & x\cos y & 3y \end{vmatrix} \;=\; \left(3\,\mathbf{i} + 5x^2 e^{5z}\,\mathbf{j} + \cos y\,\mathbf{k}\right)$$

So we have to evaluate $\displaystyle\iint_D \left(3\,\mathbf{i} + 5x^2 e^{5z}\,\mathbf{j} + \cos y\,\mathbf{k}\right) \cdot d\mathbf{A}$.

This integral can be evaluated geometrically. The vector $d\mathbf{A}$ for the disc points in the positive x direction. (Stokes' theorem uses the right-hand rule: if you curl the fingers of your right hand in the direction of C, then your thumb points in the direction of $d\mathbf{A}$.) So:

$$\left(3\,\mathbf{i} + 5x^2 e^{5z}\,\mathbf{j} + \cos y\,\mathbf{k}\right) \cdot d\mathbf{A} \;=\; 3\,dA$$

Therefore:

$$\iint_D \left(3\,\mathbf{i} + 5x^2 e^{5z}\,\mathbf{j} + \cos y\,\mathbf{k}\right) \cdot d\mathbf{A} \;=\; \iint_D 3\,dA \;=\; 3 \times (\text{the area of the disc}) \;=\; 12\pi \qquad \blacksquare$$

EXAMPLE 2 Let S be the surface $z = x(1-x)y(1-y)$ for $0 \le x \le 1$ and $0 \le y \le 1$. Evaluate the integral $\displaystyle\iint_S x\mathbf{k} \cdot d\mathbf{A}$, where $d\mathbf{A}$ is the upward-pointing normal vector.

SOLUTION If we wish to use Stokes' theorem, we must express $x\mathbf{k}$ as the curl of some vector field $\mathbf{F}$. The formula for the curl is:

$$\operatorname{curl}\mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \;=\; \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\; \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\; \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

So what we need is:

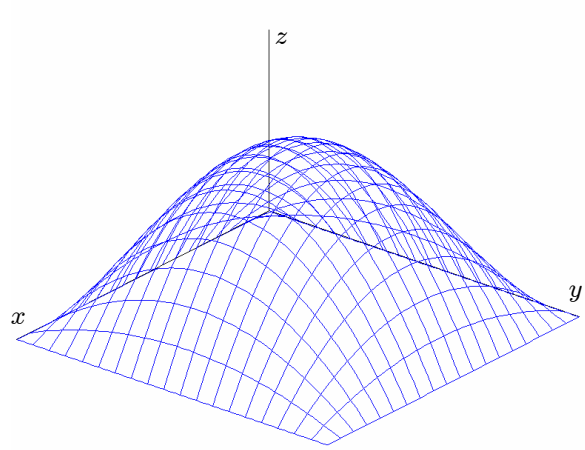
$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0 \qquad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0 \qquad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = x$$

It is not hard to guess that $F_y = \frac{1}{2}x^2$, with $F_x = F_z = 0$. Indeed:

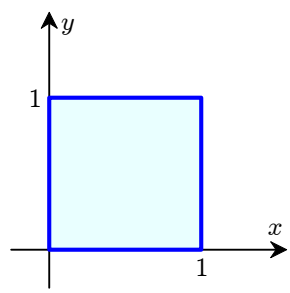
$$\operatorname{curl}\left(\frac{1}{2}x^2\,\mathbf{j}\right) \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{1}{2}x^2 & 0 \end{vmatrix} \;=\; x\mathbf{k}$$

By Stokes' Theorem, we conclude that $\displaystyle\iint_S x\mathbf{k} \cdot d\mathbf{A} \;=\; \oint_C \frac{1}{2}x^2\,dy$, where C is the boundary curve of the surface S.

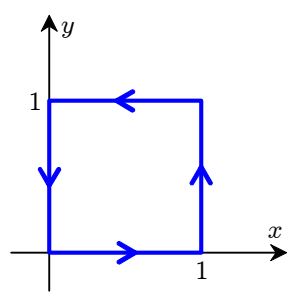
So what is the boundary of S? Well, the equation $z = x(1-x)y(1-y)$ specifies a surface whose z-coordinate varies with horizontal position:



The allowed values of x and y are determined by the inequalities $0 \le x \le 1$ and $0 \le y \le 1$, which describe a square on the xy-plane:



The sides of this square are along the lines $x = 0$, $y = 0$, $x = 1$, and $y = 1$. Looking at the equation, we see that $z = 0$ for these values of x and y, so the boundary of the surface is just the boundary of the square on the xy-plane:



We can evaluate $\oint_C \frac{1}{2}x^2\,dy$ geometrically:

$$\oint_C \frac{1}{2}x^2\,dy = \int_{\text{top}} \frac{1}{2}x^2\,dy + \int_{\text{bottom}} \frac{1}{2}x^2\,dy + \int_{\text{left}} \frac{1}{2}x^2\,dy + \int_{\text{right}} \frac{1}{2}x^2 dy$$

The top and bottom sides are horizontal, so $dy = 0$. Furthermore, $x = 0$ along the left edge, and $x = 1$

along the right edge, so:

$$\oint_C \frac{1}{2}x^2\,dy \;=\; 0 + 0 + 0 + \int_{\text{right}} \frac{1}{2}\,dy \;=\; \frac{1}{2}$$

■

EXAMPLE 3 Let S be the upper hemisphere of the unit sphere $x^2 + y^2 + z^2 = 1$. Use Stokes' theorem to evaluate $\iint_S \left(x^3e^y\,\mathbf{i} - 3x^2e^y\,\mathbf{j}\right) \cdot d\mathbf{A}$, where $d\mathbf{A}$ is the upward-pointing normal vector.

SOLUTION If we wish to use Stokes' theorem, we must express $x^3e^y\,\mathbf{i} - 3x^2e^y\,\mathbf{j}$ as the curl of some vector field $\mathbf{F}$. The formula for the curl is:

$$\operatorname{curl}\mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \;=\; \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\; \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\; \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

So what we need is:

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = x^3e^y \qquad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = -3x^2e^y \qquad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

If we guess that F_y and F_x are zero, it is not too hard to figure out that $F_z = x^3e^y$. Indeed:

$$\operatorname{curl}\left(x^3e^y\,\mathbf{k}\right) \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & x^3e^y \end{vmatrix} \;=\; x^3e^y\,\mathbf{i} - 3x^2e^y\,\mathbf{j}$$

By Stokes' Theorem, we conclude that $\iint_S \left(x^3e^y\,\mathbf{i} - 3x^2e^y\,\mathbf{j}\right) \cdot d\mathbf{A} \;=\; \oint_C x^3e^y\,dz$, where C is the boundary curve of the surface S.

Since S is the upper hemisphere of the unit sphere, C is just the unit circle on the xy-plane. By the right-hand rule, C is oriented counterclockwise. (It would be clockwise if we had started with the downward-pointing $d\mathbf{A}$.) Then $\oint_C x^3e^y\,dz$ is zero, since z is not changing over the course of the circle. We conclude that $\iint_S \left(x^3e^y\,\mathbf{i} - 3x^2e^y\,\mathbf{j}\right) \cdot d\mathbf{A} = 0$. ■

EXAMPLE 4 Let S be the surface defined by $z = x^2 + y^2$ for $z \le 4$. Use Stokes' Theorem to evaluate $\iint_S \left(-3xz^2\,\mathbf{i} + z^3\,\mathbf{k}\right) \cdot d\mathbf{A}$, where $d\mathbf{A}$ is the upward-pointing normal vector.

SOLUTION If we wish to use Stokes' theorem, we must express $-3xz^2\,\mathbf{i} + z^3\,\mathbf{j}$ as the curl of some vector field $\mathbf{F}$. The formula for the curl is:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

So what we need is:

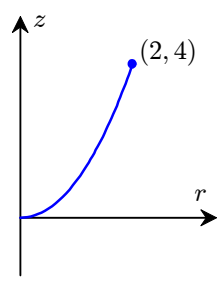
$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = -3xz^2 \qquad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0 \qquad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = z^3$$

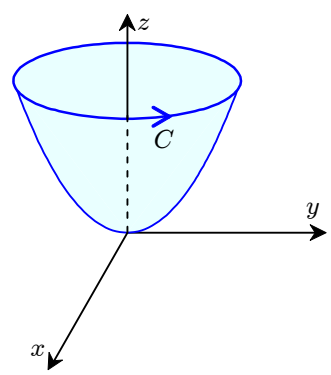
If we guess that F_x and F_z are zero, it is not hard to figure out that $F_y = xz^3$. Indeed:

$$\operatorname{curl}\left(xz^3\,\mathbf{j}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xz^3 & 0 \end{vmatrix} = -3xz^2\,\mathbf{i} + z^3\,\mathbf{k}$$

By Stokes' Theorem, we conclude that $\iint_S \left(-3xz^2\,\mathbf{i} + z^3\,\mathbf{k}\right) \cdot d\mathbf{A} = \oint_C xz^3\,dy$, where C is the boundary curve of the surface S.

So what is the boundary of S? Well, the equation for the surface S can be expressed as $z = r^2$, where $z \le 4$. This appears as a parabola on the rz-plane:





Therefore, the surface S is a bowl-shaped paraboloid. Its boundary curve C is a counterclockwise circle on the plane $z = 4$ with radius 2, with parameterization:

$$\begin{aligned} x &= 2\cos t \\ y &= 2\sin t \qquad 0 \le t \le 2\pi \\ z &= 4 \end{aligned}$$

(The circle is counterclockwise by the right-hand rule.) Thus:

$$\begin{aligned}
\oint_C xz^3\,dy &= \int_0^{2\pi} (2\cos t)(4)^3(2\cos t\,dt) \\
&= 256\int_0^{2\pi} \cos^2 t\,dt \\
&= 256\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\,dt \qquad \text{(double-angle formula)} \\
&= 128\left[t+\frac{1}{2}\sin 2t\right]_0^{2\pi} \\
&= 256\pi
\end{aligned}$$

■