Computer simulations in Physics

Game models

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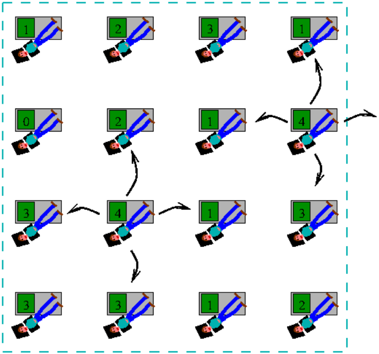
May 19, 2022

Self-Organized Criticality

- Critical state: inflection point in the critical isotherm
- Power law functions of correlation length, relaxation time
- Control parameter, generally temperature
- Critical point as an attractor?
- Why? Power law: We see many cases
  - $1/f$ noise (music, ocean, earthquakes, flames)
  - Lack of scales (market, earthquakes)
- Underlying mechanism?

Bak-Tang-Wiesenfeld model

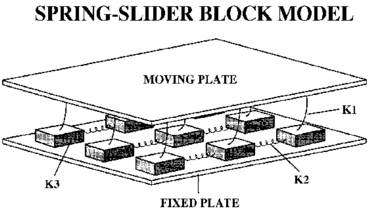
- Originally a sandpile model
- Better explained as a *Lazy Bureaucrat model*:
  - Bureaucrats are sitting in a large office in a square lattice arrangement
  - Occasionally the boss comes with a dossier and places it on a random table
  - The bureaucrats do *nothing* until they have less than 4 dossiers on their table



- Once a bureaucrat has 4 or more dossiers on its table starts to panic and distributes its dossiers to its 4 neighbors
- The ones sitting at the windows give also 1 dossier to its neighbors and throw the rest out of the window.

Bak-Tang-Wiesenfeld model

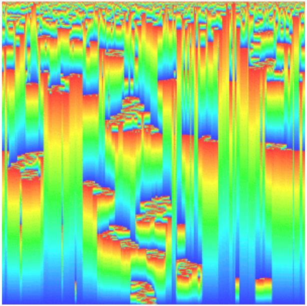
- Originally a sandpile model
- Better explained as a *Lazy Bureaucrat model*:
- Best application: Spring block model of earthquakes:
  - Masses sitting on a frictional plane in a grid are connected with springs to eachother and to the top plate



- Top plate moves slowly, increasing the stress on the top springs slowly and randomly
- If force is large enough masses move which increases the stress on the neighboring masses

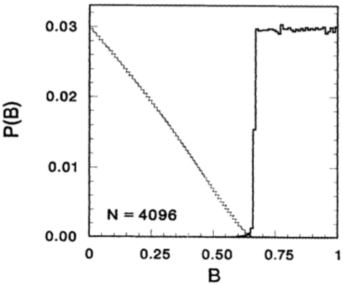
Bak-Sneppen model of evolution

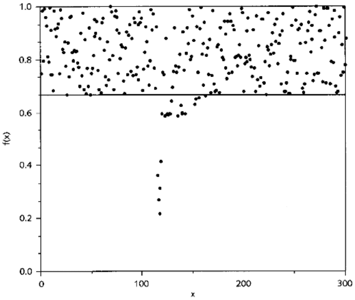
- N species all depends on two other (ring geometry)
- Each species are characterized by a single *fitness*
- In each turn the species with the lowest fitness dies out and with it its two neighbors irrespective of their fitness
- These 3 species are replaced by new ones with random fitness
- Inital and update fitness is uniform between $[0, 1]$



Bak-Sneppen model of evolution: Results

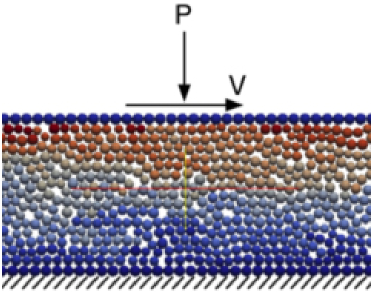
- Steady state with avalanches
- Avalanches start with a fitness $f > f_c \simeq 0.66$
- Avalanche size distribution power law
- Distance correlation power law

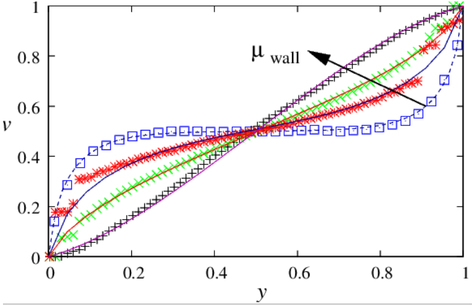




Bak-Sneppen model of evolution an application: Granular shear

- Fitness → Effective friction coefficient
- Specimen with lowest fitness dies out → block is sheared at weakest position (shear band)
- Neighbors, related species die out and replaced by new species → structure gets random around the shear band.

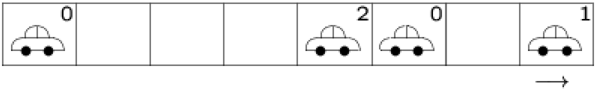




Traffic models

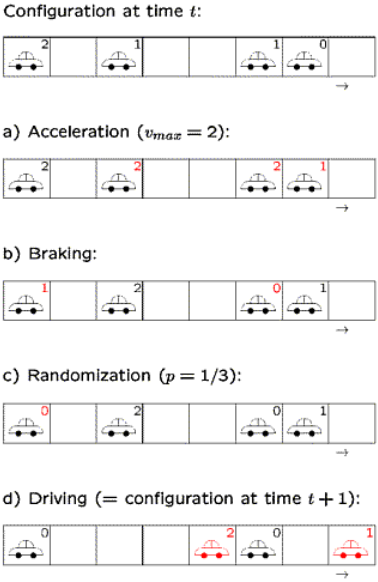
Nagel–Schreckenberg model

- Periodic 1d lattice (ring) Autobahn
- discretized in space and time
- Cars occupying a lattice moving with velocities $v = 0, 1, 2, 3, 4, 5$
- Remark, if max speed is 126 km/h, then lattice length is 7 m, a very good guess for a car in a traffic jam
- It uses parallel update: at each timestep all cars move v sites forward

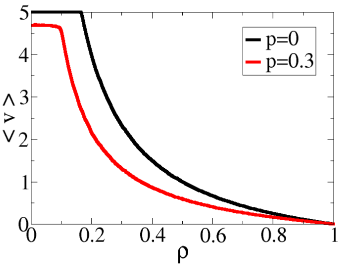


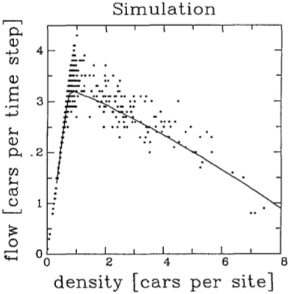
Nagel–Schreckenberg model

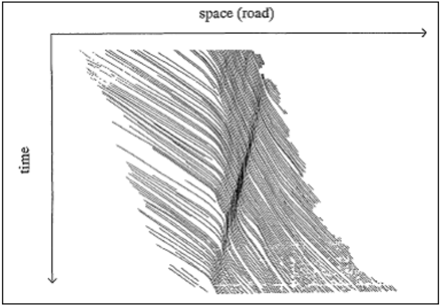
- Algorithm:
  1. **Acceleration**: All cars not at the maximum velocity increase their velocity by 1
  2. **Slowing down**: Speed is reduced to distance ahead (1 sec rule)
  3. **Randomization**: With probability p speed is reduced by 1
  4. **Car motion**: Each car moves forward the number of cells equal to their velocity.



Emergence of traffic jams







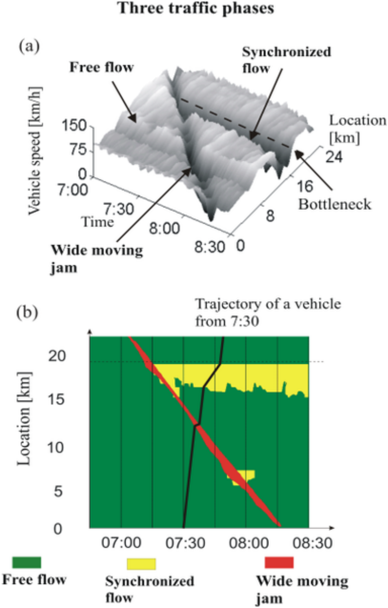
t=0

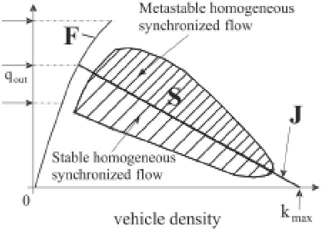
Nagel–Schreckenberg model

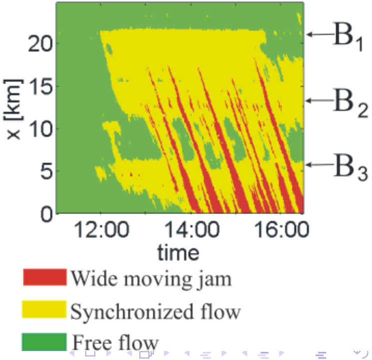
- Transition from free-flow to jammed state
- Jammed state is a phase-separated phase
- Without randomization a sharp transition
- Had been used in NRW to predict traffic jams


autobahn.NRW
Hochrechnung für Bundesautobahnen
Montag, 01.07.2013, 08:05 h
aktuelle Verkehrslage
Prognose 30 min
Prognose 60 min
+7
Langzeit Prognose
Fahrzeitprognose
Farbmodus: Standard
Baustelleninfo
Verkehrsdichten
frei
zähfließend
dicht
gestaut
Baustelle mit geringer Staugefahr
Baustelle mit hoher Staugefahr
Anschlussstelle
Sperrung der Autobahn
Vollsperrung der Anschlussstelle
Teilsperrung der Anschlussstelle
Informationen
Impressum
Version 3.6.1
0 km
15 km
Wesel
Recklinghausen
Lünen
Geldern
Kamp-Lintfort
Bottrop
Gelsenk.
Herne
Dortmund
Moers
Oberhsn.
Essen
Bochum
Duisburg
Mülheim
Witten
Kempen
Krefeld
Hagen
Velbert
Gevelsberg
Viersen
Düsseldorf
Mönchengladbach
Neuss
Wuppertal
Lüdenscheid

Three-phase traffic theory





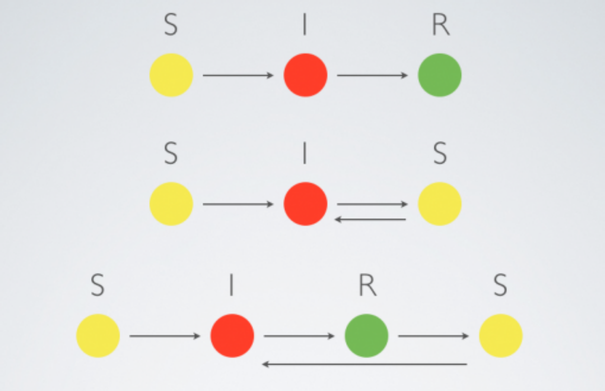


Disease spreading, SIR model

- S: susceptible (can be infected with prob. β if meets an infected)
- I: Infected (may infect susceptible, but may recover with prob. ν).
- R: Recovered (Immune to the disease)
- Other versions:
  - SI: agents do not recover (e.g. information spreading)
  - SIS: recovered people can get disease again
  - SIRS: recovered agents may become susceptible (e.g. influenza)

Disease spreading, SIR model

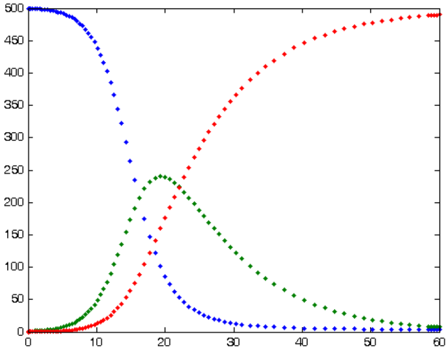
- S: susceptible
- I: Infected
- R: Recovered



SIR model, connected graph

Governing equations:

$$\dot{S} = -\beta I S$$
$$\dot{I} = \beta I S - \nu I$$
$$\dot{R} = \nu I$$



SIR model, connected graph

Governing equations:

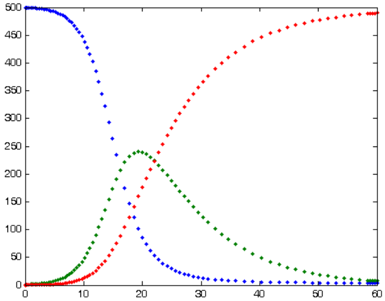
$$\dot{S} = -\beta I S$$
$$\dot{I} = \beta I S - \nu I$$
$$\dot{R} = \nu I$$

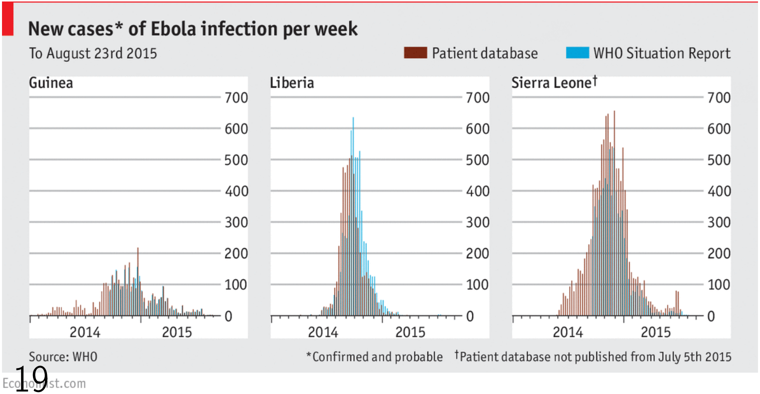
- Early stage $S \simeq 1$

$$I \simeq I_0 \exp[(\beta - \nu)t]$$

- $R = \beta/\nu$ epidemic threshold
  - $R > 1$ outbreak
  - $R < 1$ localized

SIR model vs. reality

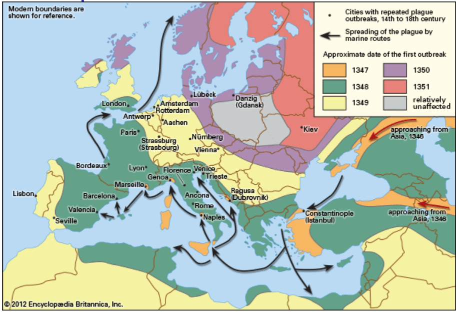


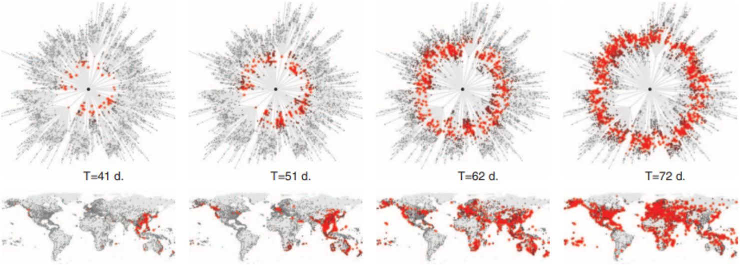


Algorithm for the SIR model

1. List of initially infected nodes is I
2. Get a random (infected) node u from the list I
3. For all neighbors w of u do 4.
4. If w is susceptible change it to infected with probability β, and enqueue it into list I
5. With probability ν change state of u to recovered otherwise put it back to I
6. If I is not empty go back to 2.

SIR on space and network





Brockmann-Helbing

Predator prey model

- $N(t)$ number of predators
- $E(t)$ number of prey
- Model (Lotka 1925, Volterra 1926):

$$\begin{aligned} \dot{E}(t) &= \beta_E E(t) - [\mu_E N(t)] E(t) \\ \dot{E}(t) &= [\beta_N E(t)] N(t) - \mu_N N(t) \end{aligned} \tag{1}$$

- Solution $\dot{E} = \dot{N} = 0$:

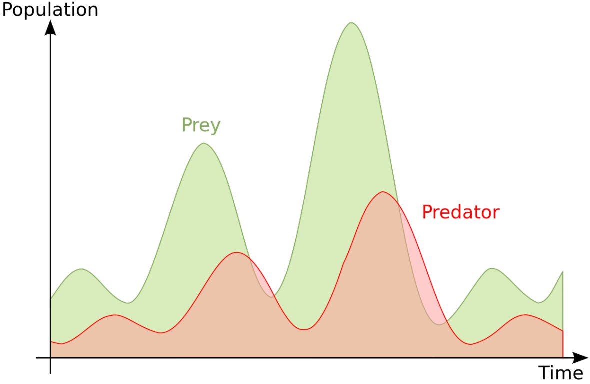
$$\begin{aligned} N = E = 0 \\ N = \beta_E / \mu_E, \quad E = \nu_N / \beta_N \end{aligned} \tag{2}$$

Predator prey model

- Solution $\dot{E} = \dot{N} = 0$:

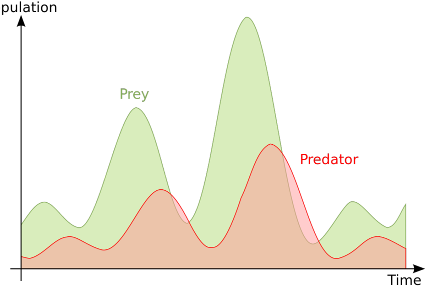
$$N = E = 0$$
$$N = \beta_E / \mu_E, \quad E = \nu_N / \beta_N \tag{3}$$

- Numerical solution:

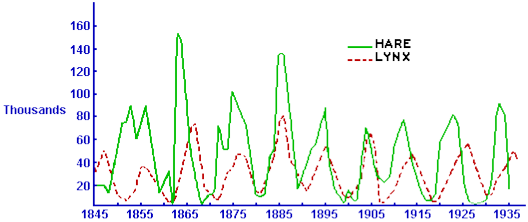


Predator prey model

- Numerical solution:



- Reality:



Other agent based models

- Agents are nodes
- Interactions through links
- Any network:
  - Lattices
  - Random networks
  - Scale-free
  - Fully connected graphs
- Examples:
  - Opinion models (not this time)
  - Minority models
  - Game models

Flocking Model

Flocking model

- Birds move with constant velocity (v_0)
- Align themselves to neighbors
- Some noise due to inaccurate averaging
- Differential equation

$$\theta_i(t + \Delta t) = \langle \theta(t) \rangle_{|\mathrm{r}_i - \mathrm{r}_j| < R} + \xi$$

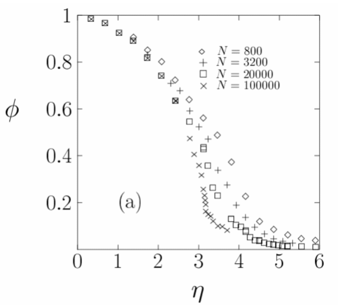
- Upgrade position:

$$\mathrm{r}_i(t + \Delta t) = \mathrm{r}_i(t) + v_0 \mathrm{e}(\theta_i(t))\Delta t$$

where $\mathrm{e}(\theta)$ is a unit vector in the direction of θ.

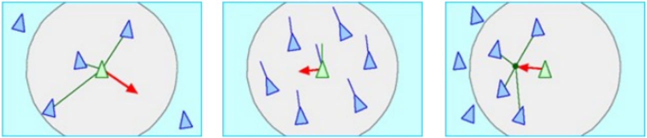
Flocking model

- Birds move with constant velocity (v_0)
- Align themselves to neighbors
- Some noise due to inaccurate averaging
- Phase diagram 1d:



Flocking model

- Birds move with constant velocity (v_0)
- Align themselves to neighbors
- Some noise due to inaccurate averaging
- Non-physicist model:



Separation: steer to avoid crowding local flockmates

Alignment: steer towards the average heading of local flockmates

Cohesion: steer to move toward the average position of local flockmates

http://www.red3d.com/cwr/boids/

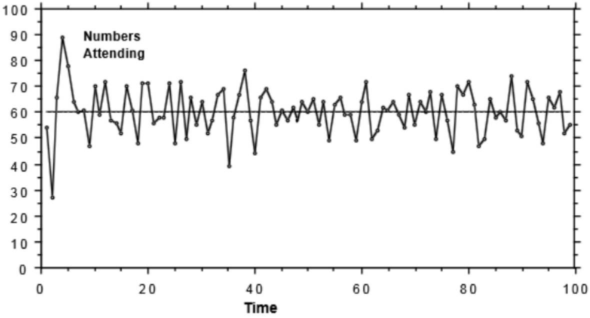
Minority models

"It is not worth an intelligent man's time to be in the majority. By definition, there are already enough people to do that."
Godfrey Harold Hardy
"Csak a döglött hal úszik az árral" - "Only dead fish swim with the tide"

- El Farol Bar problem
- Irish Music Thursdays
- Music is unenjoyable if more than 60 people go

Minority models

- El Farol Bar problem
- Irish Music Thursdays
- Music is unenjoyable if more than 60 people go
- After a transient attendance fluctuates around 60%
- In late stages regularities (cycles) are *arbitraged* away



Memory

- Intentionalism: I know that he know that I know what he ...
- Intelligent animals: 2 levels
- Children: 2 levels
- Strong authists: 1 level
- Humans 5-7 levels

El Farol problem, strategy

- Attendance was: 44 78 56 15 23 67 84 34 45 76
- Should I stay or should I go now?
- N agents with strategies
- Agents change their strategy with respect to performance
- Similar problems:
  - Traffic decisions
  - Animals food/water
  - Shopping times

Minority model

- N players (odd for simplicity)
- Action of player i at time t is $a_i(t) = \{+1, -1\}$
- Total action: $A(t) =\sim_i a_i(t)$
- Payoff: $p_i(t) = -a_i(t)g[A(t)]$, $g(a)$ is an odd function, e.g. $\mathrm{sign}(x)$
- Information: $W(t+1) = g[A(t)] = \mathrm{sign}[A(t)]$
- Memory: limited to the last m values of W
- Strategy: A table from the 2^m possible inputs to the corresponding output

Minority model: strategy

- Memory: limited to the last m values of W
- Strategy: A table from the 2^m possible inputs to the corresponding output

input			output
-1	-1	-1	-1
-1	-1	$+1$	$+1$
-1	$+1$	-1	$+1$
-1	$+1$	$+1$	-1
$+1$	-1	-1	$+1$
$+1$	-1	$+1$	-1
$+1$	$+1$	-1	$+1$
$+1$	$+1$	$+1$	-1

Random strategy

- Having N agents, the probability of having n +1 follows a binomial distribution

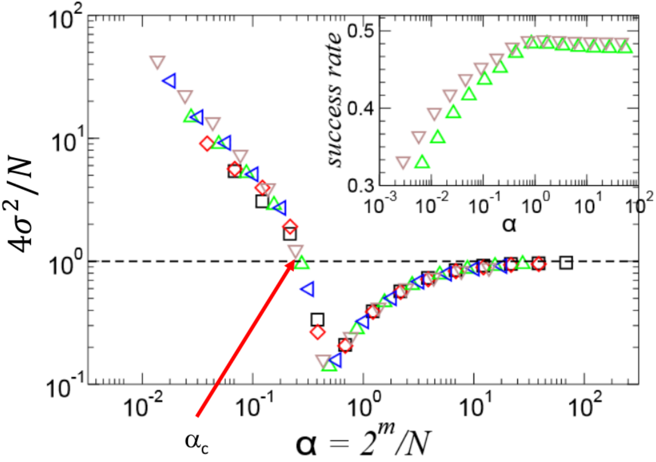
$$P(n) = \binom{N}{n} p^n (1-p)^{N-n}$$

- Average: $\langle n \rangle = pN$, $\langle n \rangle (p = 1/2) = N/2$
- Variance: $\sigma^2 = Np(1-p)$, $\sigma^2(p = 1/2) = N/2$
- Minority game:
- Average: $\langle A(t) \rangle = 0$
- Variance: σ^2/N is function of $\alpha = 2^m/N$ with

$$\lim_{\alpha \to \infty} \sigma^2/N = 1/4$$

  So the strategy with infinite memory becomes random
- At low values of α the variance increases as a power law $\sigma^2/N \sim \alpha^{-1}$

Minority model: variance



N= 101, 201, 301, 501, 701 (□, ♦, △, ▷, ▽)

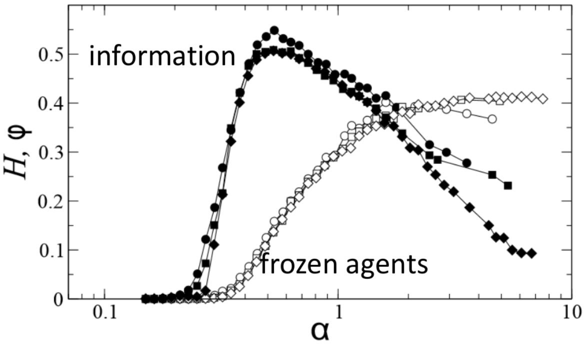
Minority model: order parameter

- Can we predict the sign of $A(t)$?
  - $\alpha < \alpha_c$: No, we have not enough information, agents are random
  - $\alpha > \alpha_c$: Yes, strong dependence, in market this can be exploited (arbitrage)
- Order parameter: information

$$H = \frac{1}{2^m} \sum_{\nu} \langle W(t+1) | W(t) = \nu \rangle$$

Minority model: variance

- Can we predict the sign of $A(t)$?
  - $\alpha < \alpha_c$: No, we have not enough information, agents are random
  - $\alpha > \alpha_c$: Yes, strong dependence, in market this can be exploited (arbitrage)

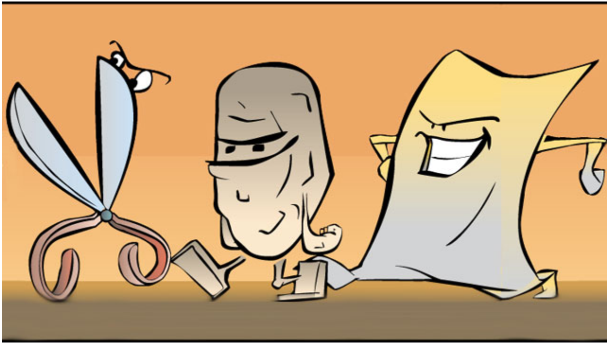


Game models:

- Rock-paper-scissors
- Prisoner's dilemma
- Chicken, hawk-dove game

Rock-paper-scissors

- No winning strategy on (truly) random opponent
- E.g bacterian and antibiotics in mice
- Grass-rabbit-fox
- Popular in games



Prisoner's Dilemma

- Two people playing the game
- Two options: Cooperate, Defect
- Cooperate: Confess the crime
- Defect: deny the crime
- Result: years in prison

	Cooperate	Defect
Cooperate	-1, -1	-3, 0
Defect	0, -3	-2, -2

Prisoner's Dilemma

- Payoff matrix
- Reward for actions based on other player's actions

	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

	Cooperate	Defect
Cooperate	1, 1	0, 2
Defect	0, 2	0, 0

Prisoner's Dilemma

- Each player with a preferred strategy that collectively results in an inferior outcome
- Dominating strategy regardless of the opponent's action
- Nash equilibrium, from which no individual player benefits from deviating

	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

	Cooperate	Defect
Cooperate	1, 1	0, 2
Defect	0, 2	0, 0

Prisoner's Dilemma

- One game $\rightarrow$ defect
- Fixed number of games $\rightarrow$ defect

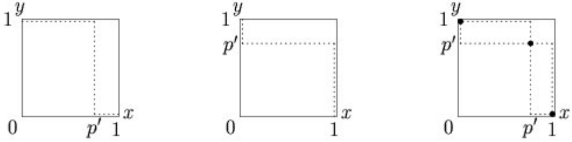
Chicken game, Hawk-Dove game

Chicken game, Hawk-Dove game

- No preferred strategy
- The best strategy is to anti-coordinate with your opponent

	Cooperate	Defect
Cooperate	0, 0	-1, 2
Defect	2, -1	-5, -5

- Example: Cold war
- Solution: anti-correlated pure strategy
- Probabilistic, or mixed strategy (play Hawk with p')



Chicken game, Hawk-Dove game difference to Prisoner's dilemma

	Cooperate	Defect
Cooperate	Reward	S, T
Defect	T, S	Punish

Hawk-Dove

	C	D
C	2, 2	1, 3
D	3, 1	0, 0

Prisoner's dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1,1

- Prisoner's dilemma:
  Temptation(T)>Reward(R)>Punish(P)>Sucker(S)
- Chicken game:
  Temptation(T)>Reward(R)>Sucker(S)>Punish(P)

Stag game

Prisoner's dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1,1

Hawk-Dove

	C	D
C	2, 2	1, 3
D	3, 1	0, 0

Stag game

	C	D
C	3, 3	0, 2
D	2, 0	1,1

- Prisoner's dilemma:
  Temptation(T)>Reward(R)>Punish(P)>Sucker(S)
- Chicken game:
  Temptation(T)>Reward(R)>Sucker(S)>Punish(P)
- Stag game:
  Reward(R)>Temptation(T)>Punish(P)>Sucker(S)

Prisoner's dilemma: multiple agents

- Against all others
- Against itself
- Against a fully random agent
- Number of agents: 14, 62

Prisoner's dilemma: multiple agents: Strategies

- Strategies for repeated games in Axelrod's tournament (1980):
- AllD: choosing D always (unconditional defector, the bad guy, ...)
- AllC: choosing C always („the good guy" or sucker)
- Random: chooses D or C with probabilities q or (1-q)
- TFT (Tit-for tat): chooses C first, then she repeates/reciprocates the previous strategy of the co-player
- Generous TFT: TFT, but chooses C (instead of D) with a probability q
- WSLS (win-stay-lose-shift): first C or D, then she changes it if her payoff is smaller than an aspiration level (Ux<a)
- Stochastic reactive strategies: Chooses C or D with probabilities dependent on the previous decision of the co-player
- Stochastic reactive strategies with longer memory: Etc.
- Go-by-Majority cooperates on the first round, then takes majority strategy.
- ...and many more

Multiple agents: Winning strategy

- The winner is: Tit-for-tat!
- Human law
- Note that Common good was not included
- Why not "always defect"(AD), which is the Nash equilibrium of the
- Prisoners' dilemma for any finite number of plays?
- Nash equilibrium means that AS is the best strategy against AD
- AS is not dominant strategy
- It is not the best strategy for all strategies

Multiple agents: Best strategy

- Large pool of players (movie):
- It can be shown that for a repeated PD game there is no best strategy for all possible strategies
- But for a good strategy it has to be
  - Nice (do not defect first)
  - Punish others for being nasty
  - Forgive fast
  - Be efficient against yourself