(Lecture 2), 2018

Scaling and criticality:

Landau:

$$\frac{F_{cond}}{V} = \frac{1}{2}\tilde{a}(T-T_c)\cdot M^2 + \frac{1}{4}\tilde{b}M^4 - M\cdot B$$

$$\Rightarrow M(B,T): \quad \tilde{a}\cdot(T-T_c)M + \tilde{b}M^3 - B = 0$$

statement $\rightarrow$ $$M = |T-T_c|^{1/2}\cdot q_\pm\left(\frac{B}{|T-T_c|^{3/2}}\right) \qquad (1)$$

$\uparrow$ $T \gtrless T_c$

$$\Rightarrow F(T,B) = F_{cond}(M(B,T),B,T)$$

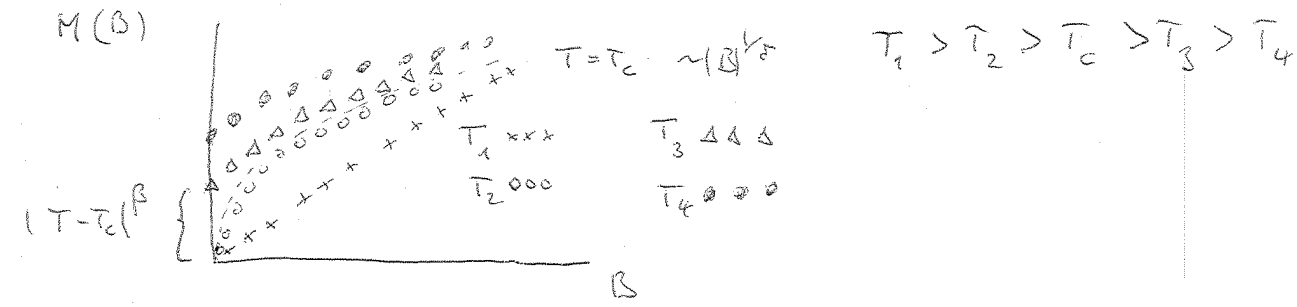
$$\frac{1}{V}F = -(T-T_c)^2 f_\pm\left(\frac{B}{|T-T_c|^{3/2}}\right) \qquad (2)$$

(1) and (2) are related:

$$\frac{1}{V}\frac{\partial F}{\partial B} = \frac{1}{V}\underbrace{\frac{\partial F_{cond}(M,B,T)}{\partial M}}_{=0}\frac{\partial M}{\partial B} + \frac{1}{V}\underbrace{\frac{\partial F_{cond}(M,B,T)}{\partial B}}_{-M} = -M(B,T)$$

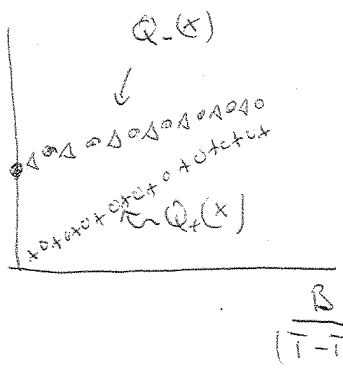
$$(2) \Rightarrow \quad q_\pm(x) = +f_\pm'(x)$$

Test (1) experimentally!



$\exists!$ β and Δ such that

$$\Rightarrow \frac{M}{|T-T_c|^\beta}$$



$$\boxed{M = |T-T_c|^\beta\, Q_\pm\left(\frac{B}{|T-T_c|^\Delta}\right)}$$

Conclusion:

$$\boxed{F/V = -|T-T_c|^{\beta+\Delta}\,\phi_\pm\left(\frac{B}{|T-T_c|^\Delta}\right)+\dots} \qquad (*)$$

exponents δ and γ:

γ: $\quad M \sim |T-T_c|^\beta\, Q_\pm\left(\frac{B}{|T-T_c|^\Delta}\right)$

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} \sim |T-T_c|^{\beta-\Delta}\cdot Q'_\pm(0) \Rightarrow \underline{\gamma = \Delta-\beta}$$

$$\Delta = \beta+\gamma$$

δ: tricky $\quad M(T=T_c,B) \sim B^{1/\delta}$

$$\Rightarrow \quad Q_\pm(x) \sim x^{\frac{\beta}{\beta+\gamma}} \qquad \text{if } x \gg 1$$

then $\quad M \sim B^{\frac{\beta}{\beta+\gamma}} \quad \Rightarrow \frac{1}{\delta} = \frac{\beta}{\beta+\gamma} \Rightarrow \delta = 1+\frac{\gamma}{\beta}$

(1) $\quad \boxed{\beta(\delta-1)=\gamma}$ $\quad$ Widom

α: $\quad F/V \sim |T-T_c|^{2\beta+\gamma}$

$$S \sim -\frac{\partial F}{\partial T} \sim |T-T_c|^{2\beta+\gamma-1}$$

$$C_V = T\frac{\partial S}{\partial T} \sim T_c\,|T-T_c|^{2\beta+\gamma-2} \sim |T-T_c|^{-\alpha}$$

$$\alpha = 2-(2\beta+\gamma) \Rightarrow \boxed{\alpha+2\beta+\gamma=2} \qquad (2)$$

Rushbrooke

(1) and (2) verified experimentally!

Derivation of (*) Wilson (renorm. group...)

Taking a closer look:

<u>Correlations</u>:

are remote spins related or independent ??

$\langle \sigma_{\underline{R}} \sigma_{\underline{R}'} \rangle$ independent $\rightarrow = \langle \sigma_{\underline{R}} \rangle \cdot \langle \sigma_{\underline{R}} \rangle$

flip together $\rightarrow \neq \langle \sigma_{\underline{R}} \rangle \cdot \langle \sigma_{\underline{R}} \rangle$

$$C(\underline{R}, \underline{R}') = C(\underline{R} - \underline{R}') \equiv \langle \sigma_R \sigma_{R'} \rangle - \langle \sigma_R \rangle \langle \sigma_{R'} \rangle$$

in a real magnet $\Rightarrow \langle M(\underline{r}) M(\underline{r}') \rangle - M^2$

we can measure that by neutrons

paramagnet (ferro) $\rightarrow$ remote spins are indep. !

$$C(\underline{r}) \sim e^{-r/\xi} \quad \text{for } r \text{ large}$$

correlation length

experiment:

$$\xi \sim |T - T_c|^{-\nu}$$

$$T = T_c \longrightarrow C(\underline{r}) \sim \frac{1}{|\underline{r}|^{D-2+\eta}}$$

$$\boxed{C(\underline{r}) \propto \frac{e^{-r/\xi}}{|\underline{r}|^{D-2+\eta}}}$$

- spins within a sphere of radius ξ move togather

- mean field theory (G.-L.) $\Rightarrow$ $\nu_{MF} = \frac{1}{2}$; $\eta_{MF} = 0$

Claim: (*) $\boxed{\chi = \frac{1}{k_B T} \int d^D \underline{r} \; C(\underline{r})}$ general theorem !

$$\Rightarrow \quad \chi \sim \int d^D \underline{r} \; \frac{e^{-r/\xi}}{|\underline{r}|^{D-2+\eta}} = \xi^{2-\eta} \sim |T-T_c|^{-\nu(2-\eta)}$$

$$\underline{r} \to \frac{r}{\xi}$$

$$\boxed{\gamma = \nu(2-\eta)}$$

mean field o.k. verified !

<u>Proof of (*)</u> : <u>linear</u> <u>response</u>

classical system $\quad H_0(\mu) \to E_\mu \leftarrow \mu$ -state

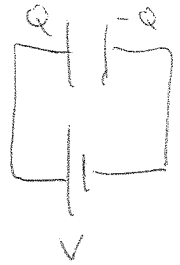
A, B physical quantities $\quad A(\mu) = A_\mu$

$$\underset{\substack{\uparrow \\ \text{equilibrium}}}{P_0(\mu)} = \frac{1}{Z} e^{-\beta E_\mu} \quad \longrightarrow \quad \langle A \rangle_0 = \sum_\mu A_\mu P_0(\mu)$$

$$Q: \quad H = H_0 - \underset{\substack{\nearrow \\ \text{"field"}}}{\mathcal{F}} \cdot B \quad \Rightarrow \quad \langle A \rangle_H = \langle A \rangle_0 + \underset{\substack{\nearrow \\ \text{generalized response function}}}{\chi_{AB}} \mathcal{F} + \mathcal{O}(\mathcal{F}^2)$$

$$\chi_{AB} = ?$$

example:



$$\longrightarrow \quad \langle Q \rangle = \underset{\substack{\uparrow \\ \chi_{QQ}}}{\mathcal{C}} \cdot V + \dots$$

$$H_0 \cong \frac{1}{2\mathcal{C}} Q^2 \quad \longrightarrow \quad H = \frac{Q^2}{2\mathcal{C}} - VQ$$

<u>Claim</u>:

$$\boxed{\chi_{AB} = \frac{1}{k_B T}\left(\langle AB \rangle_0 - \langle A \rangle_0 \langle B \rangle_0\right) = \frac{1}{k_B T} \underset{\substack{\uparrow \\ \text{correl. f.}}}{C_{AB}}}$$

<u>Proof</u>: $\quad \sum_\mu \dots \quad \Rightarrow \quad tr\{ \dots \}$

$$\langle A \rangle_H = \frac{Tr\{A e^{-\beta H}\}}{\underset{\substack{\nearrow \\ Z}}{Tr\{e^{-\beta H}\}}} = \frac{Tr\{e^{-\beta H_0} A e^{+\beta \mathcal{F} B}\} / Z_0}{Tr\{e^{-\beta H_0} e^{\beta \mathcal{F} B}\} / Z_0} \cong \frac{\langle A(1+\beta \mathcal{F} B)\rangle_0}{\langle 1+\beta \mathcal{F} B \rangle_0}$$

$$= \frac{\langle A_0 \rangle + \beta \mathcal{F} \langle AB \rangle_0}{1 + \beta \mathcal{F} \langle B \rangle_0} \approx \langle A_0 \rangle + \mathcal{F} \underbrace{\beta \left(\langle AB \rangle_0 - \langle A \rangle_0 \langle B \rangle_0 \right)}_{\chi_{AB}} + \mathcal{O}(\mathcal{F}^2) \qquad \square$$

- apply field at $\underline{R}$ $\Rightarrow B(\underline{r}) = B_0 \delta(\underline{r} - \underline{R}) \Rightarrow$ magnetization change at origin $\langle \delta M(0) \rangle = \frac{1}{k_B T} C(\underline{R}) B_0$

  homogeneous field $\rightarrow$ adds up to

$$\langle \Delta M(0) \rangle = \underbrace{\left(\frac{1}{k_B T} \int d^D \underline{R} \; C(\underline{R}) \right)}_{\chi \,!} B$$

- $\langle M(\underline{r}) M(\underline{r}') \rangle - M^2 \Leftrightarrow$ nonlocal magnetic response

  repeat derivation with $A \equiv M(\underline{r})$

  $\mathcal{F} B \Rightarrow \int d^D \underline{r}' \, M(\underline{r}') B(\underline{r}')$

$$\boxed{\langle M(\underline{r}) \rangle = \langle M(\underline{r}) \rangle_0 + \frac{1}{k_B T} \int d^3 \underline{r}' \; C_{Mr}(\underline{r} - \underline{r}') B(\underline{r}') + \dots}$$

non-local response!

- capacitor: $C = \frac{\langle Q^2 \rangle_0}{k_B T}$ ($\sim$ equipartition)

- $T \rightarrow T + \delta T$: $\beta \rightarrow \beta - \frac{1}{k_B T^2} \delta T = \beta \left(1 - \frac{\delta T}{T} \right)$

$$e^{-\beta H_0} \rightarrow e^{-\beta \left(H_0 - \frac{\delta T}{T} H_0 \right)}$$

$$\mathcal{F} \rightarrow \delta T \qquad B \rightarrow H_0 \frac{1}{T}$$

$$C_V = \frac{\partial \langle E \rangle}{\partial T} = \frac{1}{k_B T} \langle \delta H_0^2 \rangle \frac{1}{T} = \frac{1}{k_B T^2} \langle \Delta E^2 \rangle \quad \checkmark$$