Statistical physics 2, Homework 4

1. **Brownian motion, computation of $\langle \Delta x^2(t) \rangle$**
   Consider a one-dimensional Brownian motion in the absence of external force, as described by the Langevin equation:

$$\dot{v} = -\,\gamma v + \varphi(t) \ ,$$

   - Using the result, $\langle v(t)v(0)\rangle = (k_B T/m)\, e^{-\gamma|t|}$, compute the expectation value $\langle \Delta x^2 \rangle$.
   - Determine the short and long time behavior, and extract the diffusion constant D.
   - Plot the average size of displacement as a function of time $\sqrt{\langle \Delta x^2(t) \rangle}$.

2. **Diffusion equation**
   Starting from the inhomogeneous Langevin equation,

$$\dot{v} = F/m \ - \ \gamma v + \varphi(t) \ ,$$

   derive the Focker-Planck equation for $P(x,t)$.

   - Prove that $\langle \Delta x \rangle = (F/m)\, \Delta t$. Use now the fact that $\langle \Delta x^2 \rangle = 2D\, \Delta t$, and construct the corresponding coefficients $\alpha_{1,2}$ and Focker-Planck equation.
   - Show by explicit calculations that

$$G(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\bigl(-\frac{x^2}{4Dt}\bigr)$$

     is the Green's function of the diffusion equation, and satisfies

$$(\partial_t - D\, \partial_x^2)\, G(x,t) = 0 \text{ for } t > 0 \ ,$$

     and also $\lim_{t\to 0} G(x,t) = \delta(x)$.
   - Show that the general solution of the diffusion equation with some initial density, $\rho(x, t = 0) = \rho_0(x) = N P_0(x)$ is

$$\rho(x,t) = \int G(x - x', t) \rho_0(x')\, \mathrm{d}x' \ .$$

   - Now consider an abrupt density change of some solute (solved substance) in a liquid, $\rho_0(x) = \rho_0\, \theta(x)$, and compute the shape of the front as a function of time [Use the $\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2}$ function].

3. **Diffusion equation for velocity distribution**
Consider the inhomogeneous Langevin equation,

$$\dot{v} = -\,\gamma(v - \bar{v}) + \varphi(t)\ .$$

- Assuming an initial velocity, $v(0) = v_0$ for the particle, determine $\langle \Delta v \rangle$ and $\langle \Delta v^2 \rangle$ from the Langevin equation for short times, and from them the corresponding diffusion coefficients $\alpha(v)_{1,2}$, and construct the Focker-Planck equation for the velocity distribution.
- Using the relation, $S_{\phi\phi} = 2\gamma k_B T/m$ find now the stationary solution of this partial differential equation, and show that it is a thermal distribution with a drift, $P_{eq}(v) \sim \exp(-m(v - \bar{v})^2/2k_B T)$.

For fun for theorists:

4. **Diffusion equation on a finite length**
Solve the diffusion equation on a finite segment of length $2L$ using periodic boundary conditions, assuming that at time $t = 0$ we have $P_0(x) = \delta(x)$. Find the asymptotic solution for large times! [Hint: use Fourier series.]

5. **Brownian motion in structured force**
Consider the Langevin equation

$$\dot{v} = -\,\gamma v + \varphi(t)\ ,$$

but assume a structured noise, with $\langle \varphi(t)\varphi(0) \rangle = e^{-\rho|t|}\ \rho\ S_{\varphi\varphi}/2$. Determine the velocity-velocity correlation function, $C_{vv}(t)$ in this case.