Problem set 13 for Quantum Field Theory course

2019.05.14.

Topics covered

- Renormalisation of $O(N)$ theory with spontaneous symmetry breaking

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 11.1-2

In this problem set we are going to work on a single project: renormalisation in the presence of spontaneous symmetry breaking. We are working with the $O(N)$ model (slightly modified from that of Problem 10.5):

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi^k \partial^\mu \phi^k - \frac{1}{2} m^2 \phi^k \phi^k - \frac{\lambda}{4}\left(\phi^k \phi^k \phi^l \phi^l\right). \tag{1}$$

Problem 13.1 Breaking the symmetry

(a) Let us substitute $\mu^2 = -m^2$. Show that the potential function

$$V\left(\overrightarrow{\phi}\right) = -\frac{1}{2}\mu^2 \phi^k \phi^k + \frac{\lambda}{4}\left(\phi^k \phi^k \phi^l \phi^l\right) \tag{2}$$

has minima for

$$\left|\overrightarrow{\phi}\right| = \frac{\mu}{\sqrt{\lambda}} \equiv v. \tag{3}$$

Let us choose the vacuum as

$$\overrightarrow{\phi}_0 = (0, 0, \ldots, 0, v), \tag{4}$$

so the Nth component of the field is nonzero. We use perturbation theory to expand in small fluctuations around this ground state:

$$\overrightarrow{\phi} = (\pi^1(x), \pi^2(x), \ldots, \pi^{N-1}(x), v + \sigma(x)). \tag{5}$$

(b) Substitute (5) into (1) and write down the spontaneously broken Lagrangian $\mathscr{L}_M$. What is m_σ and m_π?

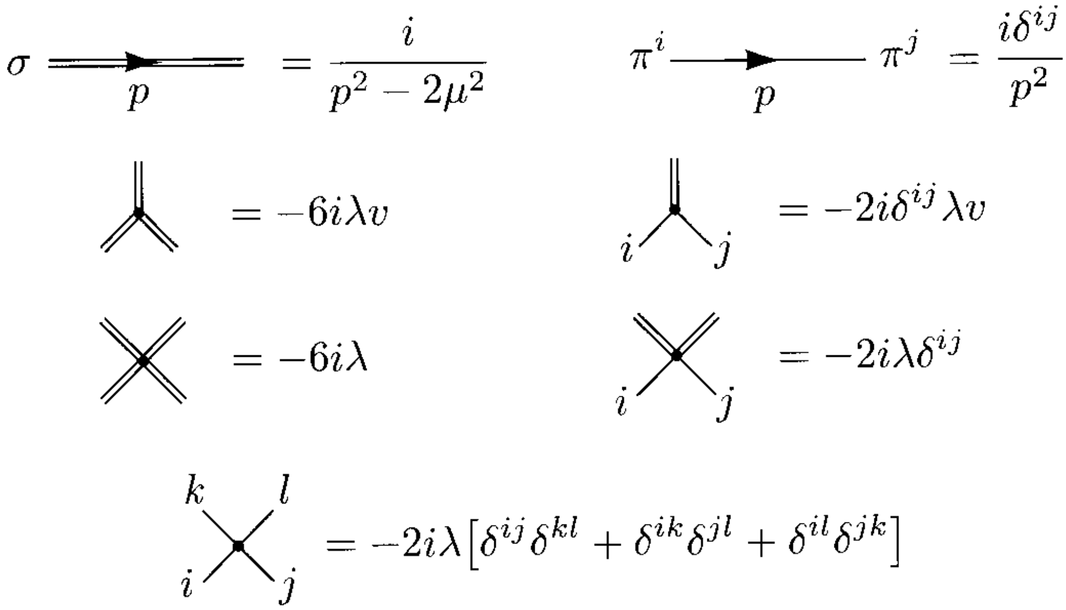
(c) The Lagrangian $\mathscr{L}_M$ has eight terms but only two parameters, μ and λ. Together with the field strength renormalisation there are only three counter terms for these parameters. Write down the counter terms for the symmetric theory $\mathscr{L}$ and then substitute (5) into them to obtain the eight counter terms as functions of δ_λ, δ_μ and δ_Z. Show that we get

$$\begin{aligned}
\mathscr{L}_M \quad = \quad & \frac{1}{2}(\partial\pi^k)^2 + \frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}2\mu^2\sigma^2 - \lambda v \sigma^3 - \lambda v (\pi^k)^2 \sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}[(\pi^k)^2]^2 + \\
& + \frac{\delta_Z}{2}(\partial\pi^k)^2 - \frac{1}{2}(\delta_\mu + \delta_\lambda v^2)(\pi^k)^2 + \frac{\delta_Z}{2}(\partial\sigma)^2 - \frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\sigma^2 - (\delta_\mu v + \delta_\lambda v^3)\sigma - \\
& - \delta_\lambda v \sigma^3 - \delta_\lambda v (\pi^k)^2 \sigma - \frac{\delta_\lambda}{4}\sigma^4 - \frac{\delta_\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\delta_\lambda}{4}[(\pi^k)^2]^2 .
\end{aligned} \tag{6}$$

Now it seems we are in trouble since there is no guarantee that all possible divergences can be renormalised using only three counter term parameters. The following exercises will provide an answer to this problem.

Problem 13.2 Renormalisation conditions and divergent graphs

(a) The Feynman rules from $\mathscr{L}_M$ are



Note that there are two kinds of propagators (π and σ).

Read off the counter term Feynman rules. Note that the symmetry factors must be handled with care; otherwise you can utilise the results of Problem 10.5.

(b) Making use of the results of Problem 12.5 show that we can write the degree of divergence for graphs in this theory (similarly to scalar ϕ^4) as

$$D = 4 - E, \tag{7}$$

where E is the number of external legs and we consider a four-dimensional model. Draw the divergent classes of graphs of the theory.

Remark: the only exception from this rule are graphs that contain 3-legged vertices since their coupling has mass dimension 1 making the graph less divergent.

(c) Let us set the renormalisation conditions. One can see from (6) that there is a divergence with one external σ leg. This is called the tadpole (can you think of a reason why?) and gives a vacuum expectation value to σ, resulting in a shift of v.

We can always parametrise our theory so that v is the exact vacuum expectation value, therefore one renormalisation condition is that this counter term should be equal to zero. To fix all counter terms we need two additional conditions: δ_Z can be fixed from the σ propagator and δ_λ from the 4σ vertex. Write down these two remaining conditions (they are similar to the "Z"-condition and the vertex condition of ϕ^4 theory, now with σ instead of ϕ)!

Remark: note that we did not set a condition for the σ mass, only for the tadpole. This means that the physical mass gets shifted by perturbation theory, so the reference point of the other two conditions changes order by order. Identify this mass shift from (6)*!*

Problem 13.3 Renormalisation of vertices

The Lagrangian (6) contains five different vertices, all of which are divergent at one-loop level. However, we have only one vertex renormalisation condition which fixes δ_λ. This exercise shows whether this is enough to renormalise all the vertices.

(a) We have a condition for the 4σ vertex. Let us begin with this one: draw the divergent one-loop graphs with 4 σ legs. Read off the divergent part of δ_λ. What is the N-dependence?

Hint: if you did not solve 13.2, you can find the necessary Feynman rules and renormalisation conditions on pp. 353-355 in PS. Note that the presence of 3-legged vertices in the loop lowers the degree of divergence. Note also that the loop integral is just the familiar $V(p^2)$ from earlier.

(b) Proceed with the $\sigma\sigma\pi\pi$ vertex. Show that the value δ_λ found above is exactly what we need to renormalise this vertex as well.

(c) Perform the calculation for the 4π vertex as well. Show that the value δ_λ found above is exactly what we need to renormalise this vertex as well.

Hint: recall the lesson learnt from Problem 10.5: when we deal with indexed legs, then the s, t, u channels show different tensor structure. However, they can be summed up due to the renormalisation conditions which yields the desired result.

(d) Finish the vertex renormalisation with the 3-legged vertices: 3σ and $\sigma\pi\pi$. Show that the same δ_λ does the job here, too.

Problem 13.4 Tadpole and propagators

After the vertices have been properly dealt with, we are left with three divergent amplitudes against two counter terms.

(a) Start with the σ tadpole. According to our conditions it should vanish exactly. Write down the terms appearing at one-loop level to rephrase this as an algebraic equation satisfied by the counter terms.

Hint: one of the loops contains a massless propagator, so this integral does not only possess a UV divergence but an infrared one as well. This can be handled by introducing an IR regulator ζ^2 in place of the mass, i.e. replace

$$\frac{i}{k^2+i\epsilon} \to \frac{i}{k^2-\zeta^2+i\epsilon}. \tag{8}$$

What is the result for $d>2$ if we remove the IR regulator by sending $\zeta \to 0$?

(b) Proceed with the σ propagator. Show that our previous conditions for δ_λ and δ_μ ensure that divergences are cancelled in this propagator as well. Show that δ_Z does not have a divergent part at one-loop level.

Hint: a part of the one-loop contributions to the σ 2-point function consists of "tadpoles" superposed on the propagator. These are exactly cancelled however due to the renormalisation conditions that stipulate that the tadpole contribution vanishes.

(c) Perform the necessary calculations for the $\pi^k\pi^l$ propagator. Focus on the divergent parts to show that they are renormalised by the already constructed counter terms.

Remark: here you also have to apply the IR regulator ζ. This involves a calculation that is not exactly identical to that of $V(p^2)$ since there are two different masses in the loop. To simplify it, you can set the external momentum to zero ($p=0$) for examination of the divergent part since the divergence does not have a p-dependence.

Problem 13.5 Goldstone theorem at one-loop order

Having removed all divergences, what remains are finite mass shifts for σ and potentially for π. However, the latter is a Goldstone boson, so its mass is expected to vanish. Naively, (6) does not guarantee that this is still true after performing one-loop renormalisation. Using the tadpole counter term,

$$\delta_\mu + v^2\delta_\lambda = -\lambda\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\left(\frac{3}{(2\mu^2)^{1-d/2}}+\frac{N-1}{(\zeta^2)^{1-d/2}}\right), \tag{9}$$

compute the finite part of the π propagator. Utilising the identity $\Gamma(x)=\Gamma(x+1)/x$, take the limit $\zeta\to 0$ to show that at zero external momentum $p=0$ the finite parts of the one-loop contributions exactly cancel, resulting in a vanishing correction to the π mass!

Hint: remember that our renormalisation conditions fix $\lambda v^2 = \mu$.