# Problem set 6 for Quantum Field Theory course

2019.03.26.

**Topics covered**

- Scattering cross-section and decay rate
- Yukawa theory and Yukawa potential
- Scattering in external electromagnetic field, Rutherford formula
- QED $e^-e^-$, $e^-e^+$ and $e^+e^+$ scattering and Coulomb interaction

**Recommended reading**

Peskin–Schroeder: An introduction to quantum field theory

- Sections 4.3-4.8

**Problem 6.1 Decay rate and scattering cross section**

The infinitesimal scattering cross section of two incoming particles $A$ and $B$ with four-momenta $p_A$, $p_B$ is expressed as

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left( \prod_f \frac{d^3\mathbf{p}_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| \mathcal{M}(p_A, p_B \to \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)} \left( p_A + p_B - \sum_f p_f \right) , \tag{1}$$

where $f$ index runs over all final states and $\mathcal{M}$ is the invariant transition amplitude. To obtain the final result for the cross section it is necessary to perform the phase space integrals to eliminate the delta functions. In the lecture, we derived the following expression for a $2 \to 2$ scattering process in the centre-of-mass frame:

$$\left( \frac{d\sigma}{d\Omega} \right)_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{CM}} |\mathcal{M}|^2 , \tag{2}$$

where $E_{CM} = E_1 + E_2$ is the total energy.

(a) Equation (1) can be slightly modified to describe the decay of an unstable particle in its rest frame:

$$d\Gamma = \frac{1}{2m_A} \left( \prod_f \frac{d^3\mathbf{p}_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| \mathcal{M}(p_A \to \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)} \left( p_A - \sum_f p_f \right) , \tag{3}$$

where $p_A = (m_A, 0, 0, 0)^T$. Following the derivation of the cross section, perform the phase spcae integrals to obtain a formula for the case of a decay to two particles in the final state:

$$\left( \frac{d\Gamma}{d\Omega} \right)_{CM} = \Theta(m_A - 2m) \frac{1}{2m_A} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{CM}} |\mathcal{M}|^2 , \tag{4}$$

where $\Theta(x)$ is the Heaviside step function and $\Omega$ is the solid angle for the direction of the final particle with momentum $\mathbf{p}_1$.

*Hint: Note that one of the integrals (say, over $\mathbf{p}_2$) is trivial due to the "spatial" part of the Dirac delta and the remaining part (energy conservation) can be used to perform the radial integral. The following identity might be useful: $pdp = EdE$.*

(b) The model discussed in Problem 5.4 exhibits such decays. Calculate the inverse lifetime $\Gamma$ of a heavy $\Phi$ particle using the above equation (we recall that the result of Problem 5.4(d) is $\mathcal{M} = -\mu$ and the mass of the decaying particle is $m_a = M$). Derive the following result:

$$\Gamma = \Theta(M - 2m)\frac{\mu^2}{8M\pi}\sqrt{1 - \frac{4m^2}{M^2}} \,. \tag{5}$$

(c) Calculate the full scattering cross-section $\sigma$ of two light $\phi$ particles with momenta $p_{A,B}$ into two light ones with momenta $p_{1,2}$ in the same model using Eq. (2) and performing the final state phase space integral. Use the result of Problem 5.4.(e):

$$\begin{aligned}\mathcal{M} =& -\mu^2\left[\frac{1}{(p_A+p_B)^2 - M^2} + \frac{1}{(p_A-p_1)^2 - M^2} + \frac{1}{(p_A-p_2)^2 - M^2}\right] \equiv \\ \equiv& -\mu^2\left[\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2}\right] ,\end{aligned} \tag{6}$$

where we introduced the Mandelstam variables $s$, $t$ and $u$.

*Hint: work in the centre-of-mass frame which allows for a simple choice of $p_A$, say, with the space-like component parallel to the $z$ axis. Express the Mandelstam variables using this setting and observe that there are only three kinds of elementary integrals to calculate.*

**Problem 6.2 Wick's theorem and Yukawa theory**

The simplified version of Yukawa's theory consists of two fields: a real scalar $\Phi(x)$ and a Dirac fermion $\Psi(x)$ with plane-wave expansions

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[c_{\mathbf{k}}e^{-ikx} + c^\dagger_{\mathbf{k}}e^{ikx}\right] , \tag{7}$$

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\sum_{s=1,2}\left[a_{s\mathbf{k}}u^s(\mathbf{k})e^{-ikx} + b^\dagger_{s\mathbf{k}}v^s(\mathbf{k})e^{ikx}\right] . \tag{8}$$

The Hamiltonian operator governing the time-evolution is

$$H = H_{\mathrm{D}} + H_{\mathrm{KG}} + \int d^3x g\overline{\Psi}\Psi\Phi \,, \tag{9}$$

where $H_{\mathrm{D}}$ is describing a free Dirac field and $H_{\mathrm{KG}}$ is the Klein–Gordon Hamiltonian of the real scalar field. We intend to treat the limit of small coupling using Wick's theorem.

(a) Calculate the normalization factor $\mathcal{N}$ of the interacting vacuum (cf. Problem 5.4) up to second order in $g$. Recall that contraction between fermionic fields only involves pairing of $\Psi$ with $\overline{\Psi}$ and also that reordering leading to fermionic pairings must use anticommutation relation of the fields.

(b) Contraction with incoming scalar particles was expressed in Problem 5.4(c)

$$\overbracket{\Phi(x)\,|\mathbf{q}}\rangle_\Phi = \Phi^{(+)}(x)\,|\mathbf{q}\rangle_\Phi = e^{-iqx}\,|0\rangle \,, \tag{10}$$

Do the same calculation involving incoming fermions and antifermions:

$$\overbracket{\Psi_\xi(x)\,|\mathbf{q}}, s, n\rangle = \Psi^{(+)}_\xi(x)\,|\mathbf{q}, s, n\rangle = u^s_\xi(\mathbf{q})e^{-iqx}\,|0\rangle \,, \tag{11}$$

$$\overbracket{\overline{\Psi}_\xi(x)\,|\mathbf{q}}, s, \overline{n}\rangle = \overline{\Psi}^{(+)}_\xi(x)\,|\mathbf{q}, s, \overline{n}\rangle = \overline{v}^s_\xi(\mathbf{q})e^{-iqx}\,|0\rangle \,, \tag{12}$$

where $n$ and $\overline{n}$ refer to particle species and the $s$ is a spin label. Similar relations for outgoing particles can be derived by taking the adjoint. Note that due to complex conjugation an outgoing antifermion must be contracted with $\Psi$, not $\overline{\Psi}$.

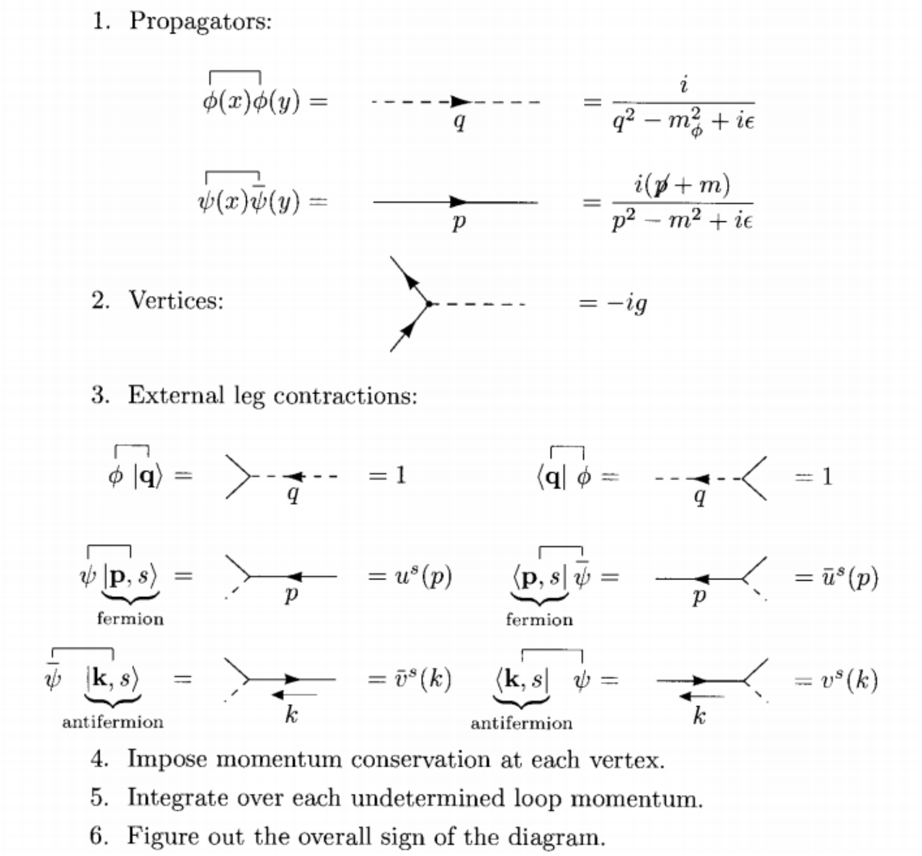


Figure 1: Feynman rules for Yukawa theory. Dashed lines denote scalar propagators, solid lines are fermionic.

(c) Use these results to compute to lowest order the amplitude of the process where a fermion and its antiparticle annihilate into a boson

$$ {}_{\Phi}\langle\mathbf{q}|\,T\exp\Big(-ig\int d^4x\overline{\Psi}(x)\Psi(x)\Phi(x)\Big)\,|\mathbf{p}_1, n; \mathbf{p}_2, \overline{n}\rangle\ , \tag{13}$$

where spin indices are dropped for brevity.

(d) Calculate the scattering amplitude of two fermions to leading order in the interaction

$$ \langle\mathbf{p}', n; \mathbf{k}', n|\,T\exp\Big(-ig\int d^4x\overline{\Psi}(x)\Psi(x)\Phi(x)\Big)\,|\mathbf{p}, n; \mathbf{k}, n\rangle\ . \tag{14}$$

As there are two fermions both in the initial and final states, a convention must be specified for the ordering of their creation and annihilation operators. Let us fix

$$ |\mathbf{p}, n; \mathbf{k}, n\rangle \propto a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}}\,|0\rangle\ , \quad \langle\mathbf{p}', n; \mathbf{k}', n| \propto \langle 0|\, a_{\mathbf{k}'} a_{\mathbf{p}'}\ . \tag{15}$$

*Hint: watch out for signs coming from exchanging fermionic fields!*

**Problem 6.3 Feynman rules for Yukawa theory, Yukawa potential**

The Feynman rules for Yukawa theory in momentum space are shown in Fig. 1 (figure from Peskin&Schroeder).

(a) Draw the lowest order Feynman diagrams for the processes (13) and (14). Note that in the latter case there are two diagrams. What is their relative sign?

(b) Now the amplitude for process (14) can be directly written in momentum space. Derive the formula for the invariant amplitude $i\mathcal{M}$.

(c) Let us consider the scattering of distinguishable non-relativistic fermions (in this case, only the first of the two diagrams contribute). Non-relativistically we can write the four-momenta as

$$p = (m, \mathbf{p})\,. \tag{16}$$

Use this expression to evaluate $(p-p')^2$ and $u^s(\mathbf{p})$ in Weyl representation up to leading order. Show by calculating the spinor products $\bar{u}^s(\mathbf{p})u^{s'}(\mathbf{p}')$ that the spin is conserved for each particle during the scattering in the nonrelativistic limit.

(d) To sum up, write the non-relativistic scattering amplitude for distinguishable particles:

$$i\mathcal{M} = \frac{ig^2}{|\mathbf{p}-\mathbf{p}'|^2 + m_\Phi^2} 2m\delta^{ss'} 2m\delta^{rr'}\,. \tag{17}$$

This is to be compared with the non-relativistic Born approximation for scattering:

$$\langle \mathbf{p}|\, iT\, |\mathbf{p}'\rangle = -i\tilde{V}(\mathbf{q}) 2\pi\delta(E_{\mathbf{p}} - E_{\mathbf{p}'})\,, \tag{18}$$

where $\tilde{V}(\mathbf{q})$ is the Fourier transform of the scattering potential and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$.
Read off $\tilde{V}(\mathbf{q})$ for the Yukawa interaction.

*Note: factors of $2m$ come from relativistic normalization and can be dropped from comparison with Born's formula.*

(e) Obtain the expression of the Yukawa potential in real space by performing an inverse Fourier transform.

*Hint: use a complex contour when calculating the radial part of this integral.*

(f) You find that two fermions feel an attractive potential due to their interaction. Without doing any lengthy calculations, just by looking at the signs argue that the situation is the same for an antifermion pair and also for a fermion-antifermion scattering.

*Hint: There is a minus sign between $\bar{u}^s(\mathbf{p})u^{s'}(\mathbf{p}')$ and $\bar{v}^s(\mathbf{p})v^{s'}(\mathbf{p}')$, and it is also necessary to disentangle all pairings properly.*

**Problem 6.4 Rutherford scattering**

Consider the scattering of a Dirac fermion on an external classical electromagnetic field described by a vector potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = e\int d^3x \overline{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x) \tag{19}$$

where $e$ is the electric charge.

(a) Use Wick's theorem to show that to leading order, the transition amplitude is given by the following expression:

$$\langle p'|\, iT\, |p\rangle = -ie\bar{u}^{s'}(\mathbf{p}')\gamma^\mu u^s(\mathbf{p})\tilde{A}_\mu(q)\,, \tag{20}$$

where $q = p' - p$ and $\tilde{A}$ is the four-dimensional Fourier transform of $A$.

(b) Assume that the external field is static so energy is conserved by such a process. Then we have

$$\langle p'|\, iT\, |p\rangle = i\mathcal{M}(2\pi)\delta(E_f - E_i)\,. \tag{21}$$

For this process (1) is modified as

$$d\sigma = \frac{1}{2E_i v_i}\frac{d^3\mathbf{p}_f}{(2\pi)^3}\frac{1}{2E_f}\left|\mathcal{M}(p_i \to p_f)\right|^2 (2\pi)\delta(E_i - E_f)\,. \tag{22}$$

where $v_i$ is the velocity of the incoming particle.
Perform the radial part of the phase space integral in $p_f$ to obtain the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2\bigg|_{p_i = p_f}\,. \tag{23}$$

*Hint: utilize the Dirac delta to impose a constraint on the relation between $\mathbf{p}_i$ and $\mathbf{p}_f$.*

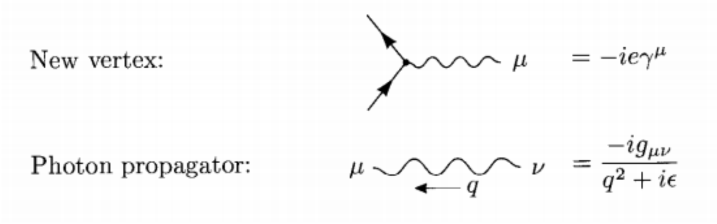


Figure 2: Feynman rules for interactions involving the photon. The metric tensor is denoted by $g_{\mu\nu}$.

(c) Now specialize the classical field to the Coulomb field of a nucleus:

$$A_0 = \frac{Ze}{4\pi r}. \tag{24}$$

with $A_i = 0$. Perform a Fourier transform to obtain $A_0(\mathbf{q})$.
Taking the nonrelativistic limit of spinors in the Weyl representation, derive the Rutherford formula

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4m^2v^4\sin^4(\theta/2)}, \tag{25}$$

where $\theta$ is the scattering angle and $\alpha = e^2/4\pi$ is the fine structure constant.

**Problem 6.5 Scattering in QED and Coulomb interaction**

In this exercise we consider scattering amplitudes of electrons and positrons. To handle such processes, two additional Feynman rules are needed for the photon that mediates the interaction (see Fig. 2.).

(a) Draw the diagrams of $e^-e^- \to e^-e^-$ scattering up to second order in $e$. What is the corresponding scattering amplitude?

(b) Consider only one of the two possible diagrams: the one without exchanging external lines. The amplitude for this process is

$$i\mathcal{M} = (-ie)^2\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p})\frac{-i\eta_{\mu\nu}}{(p-p')^2}\bar{u}(\mathbf{k}')\gamma^\nu u(\mathbf{k}). \tag{26}$$

Take the nonrelativistic limit of spinors in the Weyl representation. What is $\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p})$ in this limit?

*Hint: perform the calculation at* $\mathbf{p} = \mathbf{p}' = 0$ *and keep the term that does not vanish.*

(c) Read off $\tilde{V}(\mathbf{q})$ for the electromagnetic interaction by comparing the non-relativistic limit of (26) to expression (18). Perform an inverse Fourier transform to derive the Coulomb potential in real space. What is its sign?

*Hint: it is convenient to regularise the Fourier integral by adding a small positive term to the denominator thus reducing the integral to that found in the Yukawa case.*

(d) Draw second-order diagrams for the processes: $e^-e^+ \to e^-e^+$ and $e^+e^+ \to e^+e^+$. Write down the amplitude which involves no annihilation/external leg exchange, respectively. Compare their sign with the nonrelativistic limit of (26) to show that attraction only appears for the electron-positron pair.