Problem set 3 for Quantum Field Theory course

2019.02.26.

Topics covered

- Representation of C, P, T transformations on complex scalar field
- Trace and contraction identities of γ matrices, Fierz identities
- Field bilinears
- Locality, commutators, propagators

Recommended reading
Peskin-Schroeder: An introduction to quantum field theory

- Chapter 2
- Sections 3.4, 3.5

Problem 3.1 Dirac field bilinears

(a) Recall that

$$S(\Lambda)\gamma^\mu S(\Lambda)^{-1} = \left(\Lambda^{-1}\right)^\mu{}_\nu \gamma^\nu . \tag{1}$$

Show that $\overline{\Psi}\gamma^\mu\Psi$ is a Lorentz vector. Show that

$$\overline{\Psi}\gamma^{\mu\nu}\Psi \equiv \overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi = \frac{1}{2}\overline{\Psi}[\gamma^\mu, \gamma^\nu]\Psi \tag{2}$$

transforms as a rank 2 antisymmetric tensor.

(b) Similarly,

$$\overline{\Psi}\gamma^{\mu\nu\rho}\Psi \equiv \overline{\Psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}\Psi\,, \tag{3}$$

where $[\dots]$ denotes total antisymmetrization, transforms as a rank 3 antisymmetric tensor, and so on. In 4 dimensions, the series terminates at 4 indeces (rank 4). Show that

$$\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5\,, \qquad \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5\,. \tag{4}$$

(c) Consider the $1+4+6+4+1 = 16$ matrices

$$1\,, \quad \gamma^\mu\,, \quad \gamma^{\mu\nu}\,, \quad \gamma^\mu\gamma^5\,, \quad \gamma^5\,, \tag{5}$$

and denote them as Γ^a ($a = 1, \dots, 16$), e.g. $\Gamma^1 = 1$, $\Gamma^2 = \gamma^0$, $\Gamma^6 = \gamma^{01}$ etc.

i. Show that $(\Gamma^a)^2 = \pm 1$.
ii. Prove that for all Γ^a except $a = 1$ there exists a Γ^b such that $\{\Gamma^a, \Gamma^b\} = 0$.
iii. Prove that for all $a \neq 1$ $\mathrm{Tr}(\Gamma^a) = 0$.
(Hint: Using i. and ii. write $\mathrm{Tr}(\Gamma^a) = \pm\mathrm{Tr}\left[\Gamma^a(\Gamma^b)^2\right]$ where $\{\Gamma^a, \Gamma^b\} = 0$, and use the cyclic property of the trace.
iv. Demonstrate that for all $a \neq b$ there exists a $\Gamma^c \neq 1$ such that $\Gamma^a\Gamma^b = \Gamma^c$.

v. Finally, show that the set of Γ^a matrices is linearly independent, i.e. $\sum_a \lambda_a \Gamma^a = 0$ implies that all $\lambda_a = 0$.
(Hint: Write $0 = \mathrm{Tr}\left[\Gamma^b \sum_a \lambda_a \Gamma^a\right]$ and use iv., iii., and i.)

By this you have proved completeness of the set $\{\Gamma^a\}$ in the space of 4×4 matrices.

Problem 3.2 "Gammaology"

(a) Contraction identities

i. Show that $\gamma^\mu \gamma_\mu = 4\,\mathbb{1}$.

ii. Show that $\gamma^\mu \gamma^\nu \gamma_\mu = -2\,\gamma^\nu$.

iii. Show that $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\rho\nu}\,\mathbb{1}$.

iv. Show that $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\,\gamma^\sigma \gamma^\rho \gamma^\nu$.

v. Show that $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda \gamma_\mu = 2\left(\gamma^\lambda \gamma^\nu \gamma^\rho \gamma^\sigma + \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\lambda\right)$.

vi. Show that $\gamma^\lambda \sigma^{\mu\nu} \gamma_\lambda = 0$ and $\gamma^\lambda \sigma^{\mu\nu} \gamma^\rho \gamma_\lambda = 2\gamma^\rho \sigma^{\mu\nu}$.

(b) Trace identities

i. Using that $(\gamma^5)^2 = \mathbb{1}$ and $\{\gamma^5, \gamma^\mu\} = 0$, prove that the trace of a product of an odd number of γ matrices is zero.

ii. Show that $\mathrm{Tr}\,(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$.

iii. Using the Clifford algebra compute $\{\gamma^\mu, \gamma^\nu \gamma^\rho \gamma^\sigma\}$ and show that

$$\mathrm{Tr}\,(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\,(\eta^{\sigma\mu}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\nu}\eta^{\rho\sigma})\,. \tag{6}$$

iv. Using the notation $\not{a} = \gamma^\mu a_\mu$ we can write the previous equality as

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]\,. \tag{7}$$

Generalizing the previous identities prove that

$$\mathrm{Tr}(\not{a}_1 \ldots \not{a}_{2n}) = a_1 a_2 \mathrm{Tr}(\not{a}_3 \ldots \not{a}_{2n}) - a_1 a_3 \mathrm{Tr}(\not{a}_2 \not{a}_4 \ldots \not{a}_{2n}) + \cdots + a_1 a_{2n} \mathrm{Tr}(\not{a}_2 \ldots \not{a}_{2n-1})\,, \tag{8}$$

which allows for a recursive evaluation of such expressions.

(c) Show that

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0\,. \tag{9}$$

Prove that

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}\,. \tag{10}$$

Problem 3.3 C, P, T transformation for a complex scalar field

The plane wave expansion of a complex scalar field is

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \left[\hat{a}_{\mathbf{k}} e^{-ikx} + \hat{b}^\dagger_{\mathbf{k}} e^{ikx}\right]\,. \tag{11}$$

(a) Charge conjugation transform the annihilation operators as

$$C\hat{a}_{\mathbf{p}}C^{-1} = \xi \hat{b}_{\mathbf{p}}\,, \qquad C\hat{b}_{\mathbf{p}}C^{-1} = \tilde{\xi}\hat{a}_{\mathbf{p}}\,, \tag{12}$$

which through Eq. (11) determines the transformation of the field: $\hat{\phi}^C(\mathbf{x},t) = C\hat{\phi}(\mathbf{x},t)C^{-1}$. Compute $\left[\hat{\phi}(\mathbf{x},t), \hat{\phi}^C(\mathbf{y},t)\right]$ and show that it vanishes for space-like separation (i.e. $\hat{\phi}$ and $\hat{\phi}^C$ are local with respect to each other) if and only if $\tilde{\xi} = \xi^*$. How can we write $\hat{\phi}^C$ then?

(b) Under parity transformations,

$$P\hat{a}_{\mathbf{p}}P = \eta\hat{a}_{-\mathbf{p}}\,, \qquad P\hat{b}_{\mathbf{p}}P = \tilde{\eta}\hat{b}_{-\mathbf{p}}\,. \tag{13}$$

Similarly to the previous case, $\tilde{\eta} = \eta^*$ must hold to preserve locality. Compute $P\hat{\phi}(\mathbf{x},t)P^{-1}$.

(c) Under time reversal,

$$T\hat{a}_{\mathbf{p}}T^{-1} = \zeta \hat{a}_{-\mathbf{p}}\,, \qquad T b_{\mathbf{p}}T^{-1} = \tilde{\zeta}\hat{b}_{-\mathbf{p}}\,. \tag{14}$$

This is the same as for P, however, T is an anti-unitary operator. Show that this implies that the phases $\zeta, \tilde{\zeta}$ are not physical. (Hint: compute the eigenvalue of T on the state $e^{i\alpha}|\psi\rangle$ where $T|\psi\rangle = \zeta|\psi\rangle$ and $\alpha \in \mathbb{R}$ is arbitrary.)
Compute $T\hat{\phi}(\mathbf{x},t)T^{-1}$.

Problem 3.4 Commutation relations and locality

Recall that for the Klein–Gordon field

$$[\phi(x), \phi(y)] = \Delta_+(x-y) - \Delta_+(y-x)\,, \tag{15}$$

where

$$\Delta_+(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)}\bigg|_{p^0=E_{\mathbf{p}}}\,. \tag{16}$$

As $\Delta_+(x) = \Delta_+(-x)$ for space-like x, the Klein–Gordon commutator vanishes for space-like separations.

(a) Consider the complex scalar field in Eq. (11). Show that assuming that the creation/annihilation operators satisfy *anticommutation* relations,

$$\{a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\} = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\,, \quad \{b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}\} = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\,, \quad \{a_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}\} = 0\,, \tag{17}$$

one gets for the field anticommutator

$$\left\{\phi(\mathbf{x},t), \phi^\dagger(\mathbf{y},t)\right\} = \Delta_+(x-y) + \Delta_+(y-x)\,, \tag{18}$$

so it does not vanish for space-like separations.

(b) Repeat the calculation for $\left[\phi(\mathbf{x},t), \phi^\dagger(\mathbf{y},t)\right]$ assuming *commutation* relations,

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\,, \quad [b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\,, \quad [a_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = 0\,. \tag{19}$$

(c) Let us consider now the Dirac field

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\left[a^s_{\mathbf{k}} u^s(\mathbf{k})e^{-ikx} + (b^s)^\dagger_{\mathbf{k}}\, v^s(\mathbf{k})e^{ikx}\right]\,. \tag{20}$$

and assume that the creation/annihilation operators satisfy *commutation* relations (19) with an additional $\delta_{ss'}$. Making use of the spin sums (see Problem 2.4 c), show that

$$\left[\psi_a(\mathbf{x},t), \bar{\psi}_b(\mathbf{y},t)\right] = (i\not{\partial}_x + m)_{ab}\left[\Delta_+(x-y) + \Delta_+(y-x)\right]\,. \tag{21}$$

(d) Finally, assuming *anticommutation* relations for the mode operators (17) prove that

$$\left\{\psi_a(\mathbf{x},t), \bar{\psi}_b(\mathbf{y},t)\right\} = (i\not{\partial}_x + m)_{ab}\left[\Delta_+(x-y) - \Delta_+(y-x)\right]\,. \tag{22}$$

Problem 3.5 Dirac propagator

(a) Starting from the mode expansion of the Dirac field (20), using (17) and the spin sums (see Problem 2.4 c) derive Eq. (22) and write it as an integral over $\mathbf{p}$.

(b) Write the two terms as residues at the simple poles $p^0 = \pm E_{\mathbf{p}}$ and arrive at the expression

$$\int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2} e^{-ip(x-y)}\,. \tag{23}$$

How should the integration contour in p^0 be defined in order to obtain the retarded Green's function, $\theta(x^0-y^0)\left\{\psi_a(x), \bar{\psi}_b(y)\right\}$? What kind of contour gives the advanced Green's function, $\theta(y^0-x^0)\left\{\psi_a(x), \bar{\psi}_b(y)\right\}$?

(c) Prove that the Feynman prescription corresponds to the time ordered expectation value,

$$S_{\mathrm{F}}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip(x-y)}$$
$$= \theta(x^0-y^0)\langle 0|\psi(x)\overline{\psi}(y)|0\rangle - \theta(y^0-x^0)\langle 0|\overline{\psi}(y)\psi(x)|0\rangle \,. \quad (24)$$

Problem 3.6 Hamiltonian, momentum and angular momentum of the Dirac field

Recall that the energy-momentum tensor of the Dirac field is (see Problem 1.5 c)

$$T^{\mu\nu} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \psi = \frac{i}{2}\overline{\psi}\gamma^\mu\partial^\nu\psi - \frac{i}{2}\partial^\nu\,\overline{\psi}\gamma^\mu\psi \,. \quad (25)$$

The angular momentum tensor is (see Problem 2.2 a)

$$J^{\lambda\mu\nu} = x^\nu T^{\lambda\mu} - x^\mu T^{\lambda\nu} + \frac{1}{4}\overline{\psi}\left\{\gamma^\lambda, \sigma^{\mu\nu}\right\}\psi = x^\nu T^{\lambda\mu} - x^\mu T^{\lambda\nu} + S^{\lambda\mu\nu} \,. \quad (26)$$

(a) Write the Hamiltonian and the momentum of the field in terms of the creation/annihilation operators using Eq. (20).

(b) Write down the spin part of the conserved angular momentum $\int \mathrm{d}^3x S^{0jk}$ in terms of the creation/annihilation operators. (*Note: do not forget to normal order.*) Focusing on the z component, compute the action of $\int \mathrm{d}^3x S^{012}$ on a zero-momentum state $a_0^{s\dagger}|0\rangle$.

Problem 3.7 Fierz identities

(a) Demonstrate by explicit calculation that

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \,. \quad (27)$$

Using this show that for u_i

$$(\overline{u}_{1R}\sigma^\mu u_{2R})(\overline{u}_{3R}\sigma_\mu u_{1R}) = -(\overline{u}_{1R}\sigma^\mu u_{4R})(\overline{u}_{3R}\sigma_\mu u_{2R}) \,, \quad (28)$$

where for each i, u_{iR} 2-spinor is the lower half of a Dirac-spinor u_i.

(b) Normalize the 16 Γ^A matrices in Problem 3.1 such that

$$\mathrm{Tr}[\Gamma^A\Gamma^B] = 4\delta^{AB} \,. \quad (29)$$

(c) The general Fierz identity reads as

$$(\overline{u}_1\Gamma^A u_2)(\overline{u}_3\Gamma^B u_4) = \sum_{C,D} C^{AB}{}_{CD}\;(\overline{u}_1\Gamma^C u_4)(\overline{u}_3\Gamma^D u_2) \,, \quad (30)$$

where u_i are Dirac-spinors. The coefficients are given by

$$C^{AB}{}_{CD} = \frac{1}{16}\mathrm{Tr}[\Gamma^C\Gamma^A\Gamma^D\Gamma^B] \,. \quad (31)$$

Work out the Fierz identities for the products $(\overline{u}_1 u_2)(\overline{u}_3 u_4)$ and $(\overline{u}_1\gamma^\mu u_2)(\overline{u}_3\gamma_\mu u_4)$. Note that the Lorentz transformation properties of these expressions greatly reduce the number of coefficients that must be calculated.