Problem Set 2 for Many-body Physics II. Fall 2018

1. (15 points) Consider the photonic field in the Coulomb gauge ($\nabla \cdot \mathbf{A}(\mathbf{x}) = 0$), as specified by the vector potential:

$$\hat{\mathbf{A}}(\mathbf{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}\epsilon_0}} \left(\boldsymbol{\epsilon}^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \boldsymbol{\epsilon}_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right),$$

where

$$\boldsymbol{\epsilon}^*_{\lambda'}(\mathbf{k}) \cdot \boldsymbol{\epsilon}_\lambda(\mathbf{k}) = \delta_{\lambda\lambda'} \quad \text{és} \quad \sum_\lambda \epsilon^*_{\lambda i}(\mathbf{k}) \epsilon_{\lambda j}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2};$$

i, j are indices for the spatial components of the polarization vector. Above, we have shown the occurences of $\hbar$, but feel free to omit them later. Derive the Matsubara Green's functions of the free photonic field:

$$D^{(0)}_{ij}(\mathbf{k}, \tau - \tau') = -\langle T_\tau A_i(\mathbf{k}, \tau) A_j(-\mathbf{k}, \tau') \rangle \quad \text{and} \quad D^{(0)}_{ij}(\mathbf{k}, i\nu_n) = \int_0^\beta e^{i\nu_n \tau} D^{(0)}_{ij}(\mathbf{k}, \tau).$$

Prove that

$$D^{(0)}_{ij}(\mathbf{k}, i\nu_n) = \frac{1}{\epsilon_0} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) \frac{1}{(i\nu_n)^2 - \omega^2_{\mathbf{k}}},$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}|$.