Problem Set 1 for Many-body Physics II. Fall 2018

1. (10 points) Consider an interacting Bose/Fermi system described in the modified Heisenberg picture; the imaginary time dependence of the operators is defined using operator $\hat{K}$, which includes a term proportional to the chemical potential.

$$
\begin{aligned}
\hat{K} &= \hat{H}_0 + \hat{H}_1 - \mu \hat{N}, \\
\hat{H}_0 &= \sum_\sigma \int d^3x \overline{\psi}_{K\sigma}(\mathbf{x},\tau) \left(-\frac{\nabla^2}{2m}\right) \psi_{K\sigma}(\mathbf{x},\tau), \\
\hat{H}_1 &= \frac{1}{2}\sum_{\sigma,\sigma'} \int d^3x \int d^3x' \, \overline{\psi}_{K\sigma}(\mathbf{x},\tau)\overline{\psi}_{K\sigma'}(\mathbf{x}',\tau) V(\mathbf{x}-\mathbf{x}')\psi_{K\sigma'}(\mathbf{x}',\tau)\psi_{K\sigma}(\mathbf{x},\tau), \\
\hat{N} &= \sum_\sigma \int d^3x \overline{\psi}_{K\sigma}(\mathbf{x},\tau)\psi_{K\sigma}(\mathbf{x},\tau).
\end{aligned}
$$

   Here, $V(r)$ is any spin-independent two-body interaction. Derive the equation of motion of the Matsubara field operators, i.e., derive $\frac{\partial}{\partial\tau}\psi_{K\sigma}(\mathbf{x},\tau)$ and $\frac{\partial}{\partial\tau}\overline{\psi}_{K\sigma}(\mathbf{x},\tau)$. Ensure that the derivation is correct both for bosons and fermions. Try to give a concise formula. *Hint:* use the same-τ commutation relations several times.

2. (10 points) Show that the expectation value of any operator that can be cast into the form

$$
\hat{J} = \int d^3x \sum_{\alpha,\beta} J_{\alpha\beta}(\mathbf{x})\psi^\dagger_\alpha(\mathbf{x})\psi_\beta(\mathbf{x})
$$

   can be written as follows:

$$
-s \sum_{\alpha,\beta} \int d^3x \lim_{\mathbf{x}'\to\mathbf{x}} \lim_{\tau'\to\tau^+} J_{\alpha\beta}(x) G_{\beta\alpha}(\mathbf{x},\tau,\mathbf{x}',\tau'),
$$

   where $s=1$ for bosons, $s=-1$ for fermions. (Take care about the ordering of field operators when τ-ordering is inserted to obtain the Green's functions.)