Disorder in quantum many-body systems

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Abstract

Impurities, defects, and other types of imperfections are ubiquitous in realistic quantum many-body systems and essentially unavoidable in solid state materials. Often, such random disorder is viewed purely negatively as it is believed to prevent interesting new quantum states of matter from forming and to smear out sharp features associated with the phase transitions between them. However, disorder is also responsible for a variety of interesting novel phenomena that do not have clean counterparts. These include Anderson localization of single particle wave functions, many-body localization in isolated many-body systems, exotic quantum critical points, and "glassy" ground state phases. This brief review focuses on two separate but related subtopics in this field. First, we review under what conditions different types of randomness affect the stability of symmetry-broken low-temperature phases in quantum many-body systems and the stability of the corresponding phase transitions. Second, we discuss the fate of quantum phase transitions that are destabilized by disorder as well as the unconventional quantum Griffiths phases that emerge in their vicinity.

Contents

1. INTRODUCTION

Most real-life quantum many-body systems contain various types of random imperfections including vacancies, impurity atoms, and extended defects. Such randomness or disorder is essentially unavoidable in solid state materials as it arises naturally in the sample preparation process. Disorder has also been introduced artificially into intrinsically very clean many-body systems such as ultracold atomic gases in optical lattices.

The effects of disorder on the phases and phase transitions of quantum many-body systems are often seen in negative terms, a view that Andy Mackenzie succinctly summarized in the statement: *"For the most part, disorder in condensed matter is a pain in the neck and a barrier to truth and enlightenment"* (1). This perspective stems from the fact that random disorder can suppress new states of matter, either by preventing spontaneous symmetry-breaking or by smearing sharp features in the density of states. Moreover, disorder can round the singularities associated with phase transitions and critical points.

This review advocates for a more nuanced view: Whereas disorder can indeed do all of these negative things, it also leads to exciting, qualitatively new phenomena that do not have clean counterparts. For example, disorder can induce the spatial localization of the wave function of a quantum particle, even in the absence of interactions (2). The transition of states at the Fermi energy from extended to localized behavior is one of the possible mechanisms for metal-insulator transitions (see, e.g., Refs.(3, 4)). Building on this insight, the combined effects of disorder and interactions on transport properties have been studied extensively, leading to the identification and analysis of different universality classes of metal-insulator transitions (5, 6, 7).

In recent years, localization in disordered quantum many-body systems has reattracted enormous attention, albeit in a different context. The field of *many-body localization* deals with the very foundations of quantum statistical mechanics by exploring under what conditions an isolated quantum many-body system thermalizes. Systems that fail to quantum thermalize are many-body localized; their properties are not captured by conventional quan-

tum statistical mechanics. Reviews of this field can be found, e.g., in Refs. (8, 9, 10).

The combination of disorder and interactions can also induce novel low-temperature phases that are unique to disordered systems. These include, for example, the random-singlet phases in disordered quantum spin chains (11, 12, 13) as well as various spin glass and electric dipole glass phases in which the relevant degrees of freedom are frozen in random directions (14, 15, 16, 17).

Disorder effects in quantum many-body systems are an enormously broad area that is impossible to cover in this short review. Instead, we focus on two separate but related topics, viz., (i) the stability of clean symmetry-broken low-temperature phases and their quantum phase transitions against different types of disorder and (ii) the properties of quantum phase transitions that have been destabilized by disorder. We start by reviewing several stability criteria. They were originally derived for classical systems but have now been established, generalized, and in some cases rigorously proven for quantum systems at low temperatures. The corresponding results are scattered throughout the literature; our goal is to collect them all in one place. In the second part of this article, we review the fate of quantum phase transitions in disordered systems, and we discuss the exotic quantum Griffiths phases that emerge in their vicinity. Parts of the latter material have been reviewed in Refs. (18, 19). Here, we therefore emphasize the improved classification of critical points developed in Ref. (20) that combines and reconciles rare region effects with the Harris criterion. We also discuss recent experiments.

2. STABILITY OF PHASES AGAINST DISORDER

2.1. Symmetries and order parameters

Landau (21, 22) developed a general framework for classifying the phases in macroscopic many-body systems. Different phases can be distinguished according to their symmetries, and phase transitions generally involve the spontaneous breaking of one or more of the symmetries of the underlying Hamiltonian.[1] For example, a ferromagnetic phase breaks the global spin rotation symmetry, while the $U(1)$ symmetry associated with the phase of the macroscopic wave function is broken in a superfluid phase.

In some ordered phases, the broken symmetries include real-space symmetries. This is the case, for instance, in a charge density wave phase that spontaneously breaks the translation and rotation symmetries of the underlying solid. Other ordered phases, such as the ferromagnetic and superfluid phases mentioned above, do not break real-space symmetries but only symmetries associated with spin, phase, or other degrees of freedom. This distinction will become crucial when we introduce disorder into our system.

To quantify the degree of symmetry breaking, Landau also introduced the concept of order parameters. An order parameter is a thermodynamic quantity that is zero if the corresponding symmetry is not broken (i.e., in the disordered[2] phase), whereas it is nonzero

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[1] Currently, great research efforts are directed at phases that do not follow Landau's classification but feature unconventional topological order due to the long-range entanglement of their quantum wave functions (23, 24, 25, 26, 27, 28). The study of disorder effects on these phases and their transitions is still in its infancy and therefore not considered in this article.

[2] Unfortunately, the term "disorder" has two different meanings in the field. On the one hand, "disordered" refers to a *state* or phase without long-range (broken-symmetry or topological) order. The paramagnetic phase of a magnetic material is called a disordered phase, for example. On the other hand, "disorder" denotes randomness in the underlying *system*, i.e., the Hamiltonian.

and usually nonunique in the phase that breaks the symmetry (the ordered phase). In our example of a ferromagnetic phase, the total magnetization $\mathbf{m}$ (which is an $O(3)$ vector) is an order parameter. The order parameter for the superfluid phase is the "condensate wave function" Ψ, a complex variable. For charge density wave order with a single allowed wave vector $\mathbf{Q}$, a complex order parameter ϕ can be defined from a Fourier expansion of the charge density ρ via $\rho(\mathbf{x}) = \rho_0 + \mathbf{Re}(\phi e^{i\mathbf{Q}\cdot\mathbf{x}})$. If more than one wave vector is allowed, the order parameter becomes a complex vector.

In addition to the general framework for classifying phases, Landau put forward an approximate quantitative description, the Landau theory of phase transitions. It is based on an expansion of the free energy density f in powers of all the order parameters in the problem. In the simplest case of a single scalar order parameter m, the Landau expansion reads $f = -hm + rm^2 + vm^3 + um^4 + \ldots$ where h is the field conjugate to the order parameter. The coefficients r, v, and u can either be treated as phenomenological constants or determined from a more microscopic calculation. In general, a Landau expansion will contain all terms that are compatible with the symmetries of the system.

Within Landau theory, the order parameter is a space and time-independent constant. The theory thus contains neither the spatial inhomogeneities required for describing disorder nor the order parameter fluctuations necessary to capture the critical behavior near continuous phase transitions. This can be overcome by considering an order parameter field $m(\mathbf{x}, \tau)$ that depends on real space position $\mathbf{x}$ and imaginary time τ. The Landau free energy gets replaced by the Landau-Ginzburg-Wilson (LGW) free energy functional

$$F = \int_0^\beta d\tau \int d^d x \left[-hm(\mathbf{x}, \tau) + rm^2(\mathbf{x}, \tau) + (\nabla m(\mathbf{x}, \tau))^2 + (\partial_\tau m(\mathbf{x}, \tau))^2 + \ldots \right] \ . \qquad 1.$$

The gradient term punishes rapid changes of the order parameter; it encodes the interactions between neighboring degrees of freedom. The time derivative term controls the strengths of the quantum fluctuations. The partition function is now given by a path integral

$$Z = \int D[m(\mathbf{x}, \tau)] \exp\left(-F[m(\mathbf{x}, \tau)]\right) \ . \qquad 2.$$

Equations (1) and (2) hold in the quantum case. For classical systems, it is often sufficient to consider order parameter fields $m(\mathbf{x})$ that depend on space only. Note that the leading dynamic term in the quantum LGW functional (1) can take other forms than $(\partial_\tau m)^2$. Berry phases can produce imaginary terms (29). Moreover, if the system contains soft (gapless) excitations other than the order parameter fluctuations, the LGW functional generically features nonanalyticities that stem from integrating out these soft modes (30, 31, 32).

2.2. Types of disorder

Microscopically, disorder or randomness can have many different origins ranging from impurity atoms and vacancies to extended defects such as dislocations or grain boundaries in a crystalline solid. Thin films may experience random strains stemming from a mismatch with the substrate. Almost all disorder in condensed matter systems is time-independent over the relevant experimental time scales; this kind of disorder is called *quenched*. In contrast, *annealed* disorder changes over the time span of a typical experiment. In the present article, we almost exclusively consider quenched disorder.

In Sec. 2.1, we have seen that ordered phases can be classified according to which symmetries they break. This suggests that one should also classify the various types of disorder

according to their symmetries. Consider, for example, an Ising (easy-axis) ferromagnet in an external magnetic field $h(\mathbf{x})$ that varies randomly in space. This type of disorder is called random-field disorder; within a LGW description, it couples *linearly* to the order parameter. The corresponding term in (1) reads

$$-h(\mathbf{x})\, m(\mathbf{x}, \tau) \ . \qquad\qquad\qquad 3.$$

Random fields locally prefer a particular direction of m and therefore locally break the spin rotation symmetry. Whether or not it is broken globally depends on the distribution of $h(\mathbf{x})$. If this distribution is even, the global symmetry is preserved in a statistical sense because no direction is preferred globally.

Now consider an Ising ferromagnet containing a number of randomly distributed vacancies. Since the vacancies do not prefer a particular magnetization direction, they do not break the up-down spin symmetry of the Hamiltonian (neither locally nor globally). They cause local variations in the tendency towards ferromagnetism, i.e., they change the local critical temperature. This type of disorder is therefore called random-T_c disorder. Within a LGW theory, it couples to the square of the order parameter,[3] leading to a random variation $\delta r(\mathbf{x})$ in space of the quadratic coefficient. The quadratic term now reads

$$[r + \delta r(\mathbf{x})]\, m^2(\mathbf{x}, \tau) \ . \qquad\qquad\qquad 4.$$

Many additional kinds of disorder can appear in quantum many-body systems. For example, the disorder can consist of random phase shifts for a complex order parameter, or it can introduce easy axes in random directions in an XY or Heisenberg magnet. Moreover, strong disorder can lead to frustrated interactions that can change the thermodynamic phases qualitatively.

2.3. Imry-Ma criterion: symmetry-breaking and random-field disorder

In this section, we sketch the derivation of a criterion for the stability of a spontaneously symmetry-broken phase against random-field disorder. To be specific, consider an Ising ferromagnet subject to uncorrelated random fields that have a symmetric distribution of zero mean $[h(\mathbf{x})]_{dis} = 0$ and variance $[h(\mathbf{x})h(\mathbf{x}')]_{dis} = W\delta(\mathbf{x} - \mathbf{x}')$. In this system, the spin "up-down" symmetry is locally broken because spatial regions with positive local field h prefer a positive magnetization m while regions with negative h prefer a negative m. However, the random fields preserve the global symmetry in a statistical sense. The central question of this section is: Is global spontaneous symmetry breaking into a long-range ordered ferromagnetic state (in which the magnetization is either positive everywhere or negative everywhere) still possible?

To answer this question, Imry and Ma (33) derived a criterion for the stability of the ferromagnetic state against domain formation. Consider a system in a putative "spin-down" ferromagnetic state containing a spatial region of linear size L in which the average random field is positive and thus prefers a "spin-up" order parameter, as shown in Fig. 1a. To decide whether a "spin-up" domain forms, one needs to weigh the free energy gain due to aligning the domain with the average local random field against the free energy cost for the

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[3]In quantum field theory, the quadratic term contains the mass of the particle. Random-T_c disorder is thus also called random mass disorder.
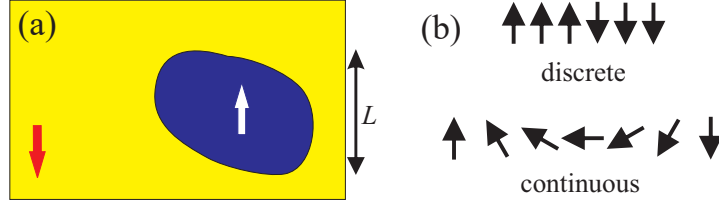
(a) The Imry-Ma criterion compares the free energy gain from aligning a domain of linear size L with the average local random field to the free energy cost for creating the domain wall. (b) For discrete order parameter symmetry, the domain wall is sharp, i.e., it has a fixed width independent of the domain size. For continuous symmetry, the change in order parameter orientation can be spread out over a length of order L.

domain wall. In d space dimensions, the domain wall is a $(d-1)$-dimensional hyper surface; its energy cost can therefore be estimated as $\Delta F_{DW} \sim \sigma L^{d-1}$ where the constant σ is the surface energy density. The energy gain from aligning the domain with the local random field is proportional to the integral of $h(\mathbf{x})$ over the domain. Estimating the typical value of this integral via the central limit theorem leads to $|\Delta F_{RF}| \sim W^{1/2} L^{d/2}$. The uniform ferromagnetic state is stable if $|\Delta F_{RF}| < \Delta F_{DW}$ for all potential domain sizes L.

For $d > 2$, ΔF_{DW} grows faster with L than $|\Delta F_{RF}|$. Thus, domains will not form if the random fields are weak, implying that the ferromagnetic state is stable. In contrast, for $d < 2$, the random field term $|\Delta F_{RF}|$ will overcome the domain wall energy ΔF_{DW} for sufficiently large L even if the random fields are weak. This means that the uniform ferromagnetic state is destroyed by domain formation.

Aizenman and Wehr (34) later proved rigorously that random field disorder prevents spontaneous symmetry breaking in dimensions $d \leq 2$ for discrete order parameter symmetry and for $d \leq 4$ in the case of continuous symmetry. The continuous symmetry case is different because the domain wall can be spread out over the entire domain (see Fig. 1b). A simple estimate of the gradient term in the LGW functional (1) yields $\Delta F_{DW} \sim L^d (\nabla m)^2 \sim L^{d-2}$ which results in a critical dimension of 4. So far, we have considered uncorrelated random fields. Long-range correlated random fields with correlations that decay as $|\mathbf{x} - \mathbf{x}'|^{-a}$ have stronger effects if $a < d$. In this case, domain formation is favored for $a < 2$ whereas the uniform ferromagnetic state is stable for $a > 2$ (35).

The Imry-Ma criterion shows that arbitrarily weak random fields prevent spontaneous symmetry breaking in $d \leq 2$. However, the length scale beyond which domains destroy the uniform state, the so-called breakup length L_B, depends sensitively on the random field strength. Comparing $|\Delta F_{RF}|$ and ΔF_{DW} yields $L_B \sim (W/\sigma^2)^{1/(d-2)}$. For the marginal dimension $d = 2$, the dependence becomes exponential, $L_B \sim \exp(\mathrm{const}/W)$, implying that domains become important only at very large scales for weak random fields.

Although the Imry-Ma criterion was originally derived for classical systems, it also applies to quantum systems at low or zero temperature. This stems from the fact that the disorder varies only in space but not in (imaginary) time. A quantum version of the rigorous Aizenman-Wehr theorem was recently proven by Greenblatt et al. (36, 37).

2.4. When do random fields emerge?

How does random-field disorder arise in realistic quantum many-body systems? To answer this question, it is crucial to distinguish order parameters that break real-space symmetries from order parameters that only break symmetries that do not involve real space.

If an order parameter does not break real-space symmetries, generic disorder does not produce random fields because it does not locally break the order parameter symmetry. For example, vacancies in a ferromagnet do not break the spin rotation symmetry. Analogously, disorder in the Josephson couplings in a Josephson junction array does not break the $U(1)$ symmetry of the superfluid order parameter. This means that the disorder does not couple to the order parameter m linearly in an LGW theory, Instead, it generically couples to m^2, i.e., it acts as random-T_c disorder.

In contrast, for order parameters that break real-space symmetries, vacancies, impurities and other defects generically generate random fields because they locally break the corresponding symmetries. For example, an electronic nematic phase spontaneously breaks the rotation symmetry of the underlying crystal lattice (38, 39, 40). Local arrangements of impurities will generally prefer a particular orientation of the nematic order, breaking its symmetry locally. They thus act as random fields and couple linearly to the order parameter in a LGW theory (41). Analogously, a charge density wave spontaneously breaks the translational symmetry. Impurities generally prefer regions of either low or high density, i.e., a particular phase of the charge density wave. Consequently, they act as random field disorder which destroys the charge density wave phase for $d \leq 4$.

Instead, the disorder induces an exotic "Bragg glass" with power-law correlations (in $d = 3$ and for weak disorder) (42, 43, 44). It has been observed, for example, in the vortex lattice of a type II superconductor (45). Recently, similar spin-density-wave and pair-density-wave glass phases have been discovered in situations where long-range spin-density-wave or pair-density-wave order is destroyed by impurities (46).

Random fields can also arise via more subtle mechanisms. $LiHoF_4$ is a dipolar Ising magnet. A magnetic field applied perpendicular to the Ising axis suppresses T_c and induces a quantum phase transition to a paramagnetic state (47). If the magnetic Ho ions are replaced by nonmagnetic Y ions in $LiHo_{1-x}Y_xF_4$, the interplay between the dilution, the off-diagonal terms of the dipolar interaction, and the applied transverse field (which breaks time-reversal symmetry) generates longitudinal random fields that qualitatively change the low-temperature behavior (48, 49, 50, 51).

2.5. Example: random-field disorder from vacancies

The diluted frustrated square-lattice Ising model with ferromagnetic nearest-neighbor interactions $J_1 > 0$ and antiferromagnetic next-nearest-neighbor interactions $J_2 < 0$ is given by

$$H = -J_1 \sum_{\langle ij \rangle} \rho_i \rho_j S_i S_j - J_2 \sum_{\langle\langle ij \rangle\rangle} \rho_i \rho_j S_i S_j \ . \qquad 5.$$

$S_i = \pm 1$ is an Ising spin, and the random variable ρ_i takes values 0 (vacancy) or 1 (occupied site) with probabilities p and $1-p$, respectively. The undiluted system features two distinct symmetry-broken phases (see Fig. 2a). For $|J_2|/J_1 < 1/2$, the low-temperature phase is ferromagnetic, but for $|J_2|/J_1 > 1/2$, the system displays stripe order characterized by a two-component order parameter $\psi_x = (1/L^2) \sum_i \rho_i S_i (-1)^{x_i}$, $\psi_y = (1/L^2) \sum_{i=1} \rho_i S_i (-1)^{y_i}$ where x_i, y_i are the coordinates of site i. The ferromagnetic phase breaks just the Z_2 Ising
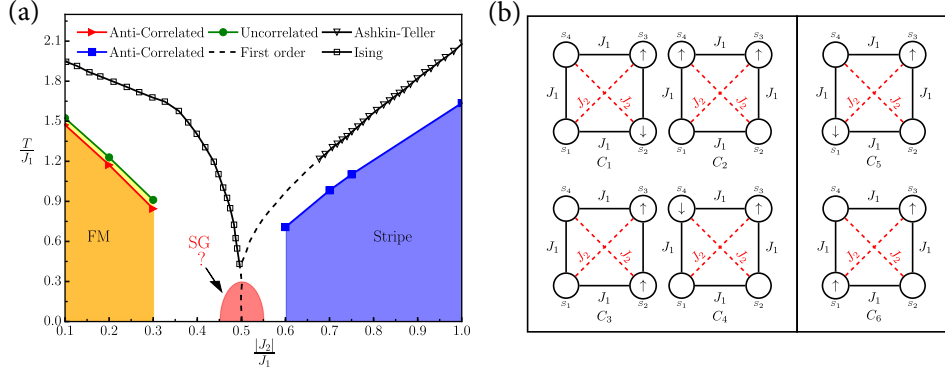
Figure 2

(a) Phase diagram of the J_1-J_2 model (5) for uncorrelated and anticorrelated vacancies of concentrations $p = 1/8$ as well as the undiluted system (open symbols). (b) Impurity configurations on 2×2 plaquettes illustrating the random-field mechanism (from Ref. (52)).

symmetry, but the stripe phase also breaks the Z_4 lattice rotation symmetry.

Since spinless impurities do not break the Ising symmetry, they do not create random fields for the ferromagnetic order parameter, the magnetization m. Instead they act as random-T_c disorder and couple to m^2. Consequently, the ferromagnetic phase is expected to survive in the presence of impurities.

Even though a single impurity does not break the Z_4 lattice rotation symmetry, spatial arrangements of more than one impurity do. If two vertical nearest neighbors are both occupied by impurities, vertical stripes have a lower energy (by $-2J_1$) than horizontal stripes (see Fig. 2b). Similarly, if impurities occupy two horizontal nearest neighbors, horizontal stripes are favored. Impurities on nearest neighbor sites thus create random fields for the nematic order parameter $\eta = \psi_x^2 - \psi_y^2$ that are expected to destroy the stripe phase.

Monte Carlo simulations with uncorrelated impurities (52) have confirmed that the stripe phase is destroyed while the ferromagnetic phase survives (see Fig. 2a). A similar mechanism was identified in an XY antiferromagnet on a pyrochlore lattice (53).

Because random fields only appear if pairs of impurities occupy nearest neighbor sites, they will be absent for perfectly anticorrelated impurities where such pairs are forbidden.[4] Monte Carlo simulations (52) indeed show that the stripe phase survives the introduction of perfectly anticorrelated disorder, see Fig. 2a. The preservation of the stripe phase by anticorrelations between impurities is analogous to the protection of clean quantum critical points by local disorder correlations in a random quantum Ising chain (55).

3. STABILITY OF PHASE TRANSITIONS AGAINST DISORDER

We now turn to the stability of phase transitions against disorder. The focus will be on random-T_c disorder because random-field disorder completely prevents symmetry-breaking

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[4]At zero temperature, this is an exact result. Entropic effects may generate random fields at nonzero temperatures (54), but they are expected to be extremely weak at low temperatures.

in $d \leq 2$. If the ordered phase survives in the presence of random fields in $d > 2$, its phase transition is usually controlled by a classical zero-temperature renormalization group fixed point (56). This means that the static random-field fluctuations dominate over both the thermal fluctuations and the quantum fluctuations. This has been explicitly demonstrated, for example, for the quantum spherical model (57) in a random field (58).

In contrast, weak random-T_c (random-mass) disorder does not affect the stability of the bulk phases, but it can destabilize the phase transitions between them. In this section, we review the corresponding stability criteria.

3.1. Imry-Ma criterion again: stability of first-order transitions against random-T_c disorder

First order phase transitions are characterized by the macroscopic coexistence of two distinct phases at the transition point. Random-T_c disorder locally favors one phase over the other. We therefore arrive at the same question as in Sec. 2.3: Will uniform macroscopic phases survive at the transition point or will the system form finite-size domains of the locally favored phase?

To answer this question, one can adapt the Imry-Ma criterion (59, 60). Consider a single domain of the first phase located in a favorable region of the random-T_c disorder and embedded in the second phase. The free energy cost of the surface increases as $\Delta F_{surf} \sim \sigma L^{d-1}$ with domain size L where σ is the surface energy density.[5] The free energy gain of the domain from being in the "right" phase is obtained from central limit theorem as $|\Delta F_{dis}| \sim W^{1/2} L^{d/2}$ where W is the variance of the random T_c disorder. Phase coexistence is therefore impossible in $d \leq 2$ for arbitrarily weak random T_c disorder. This means the first-order phase transition is destroyed. For $d > 2$, phase coexistence is possible, and the first-order transition survives for disorder strengths below a certain threshold.

Since all of these results had originally been derived for classical phase transitions, there was some uncertainty initially about their applicability to quantum phase transitions (61). However, a quantum version of the Aizenman-Wehr theorem has now been proven (36, 37). Moreover, explicit results for first-order quantum phase transitions in various types of quantum spin chains confirm the criterion (61, 62, 63, 64).

The question of what happens to a first-order transition that is destabilized by random-T_c disorder is beyond the reach of the Imry-Ma criterion. Transitions between an ordered and a disordered phase are often rounded into continuous ones. The fate of transitions between two different ordered phases is more complex because Landau's classification does not allow such transitions to be continuous.[6] Therefore, an intermediate phase often appears.

3.2. Harris criterion: stability of critical points

To derive a criterion for the stability of a clean critical point against weak random-T_c disorder, we divide the system into blocks whose size is the correlation length ξ. Because of the disorder, each block i has its own critical temperature $T_c(i)$. We now compare the variations ΔT_c of these block critical temperatures with the distance $T - T_c$ from the global

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[5]This terms scales as L^{d-1} independent of the symmetry of the order parameter within each phase because the two distinct phases are generally *not* connected via a continuous transformation.

[6]Continuous phase transitions between different ordered phases can occur within exotic scenarios such as deconfined quantum criticality (65, 66).

critical point. As long as $\Delta T_c < |T - T_c|$, all blocks are in the same phase, and the system is approximately uniform. For $\Delta T_c > |T - T_c|$, however, different blocks are on different sides of T_c, making a uniform transition impossible.

Consequently, the clean critical behavior is stable if $\Delta T_c < |T - T_c|$ remains valid as the transition is approached, i.e., for $\xi \to \infty$. Because the $T_c(i)$ of a block is determined by the average over a large number of random variables, central limit theorem predicts that $\Delta T_c \sim \xi^{-d/2}$. The global distance from criticality is related to the correlation length via $\xi \sim |T - T_c|^{-\nu}$ where ν is the clean correlation length exponent. The condition $\Delta T_c < |T - T_c|$ in the limit $\xi \to \infty$ implies Harris' exponent inequality (67)

$$d\nu > 2 \ . \tag{6.}$$

If Harris' inequality is fulfilled, the ratio $\Delta T_c / |T - T_c|$ approaches zero for $\xi \to \infty$. The system thus becomes asymptotically clean at large length scales. In contrast, if Harris' inequality is violated, $\Delta T_c / |T - T_c|$ increases as the transition is approached, destabilizing the uniform clean transition. We emphasize that the Harris criterion is a necessary condition for the stability of the clean critical point, not a sufficient one because it only tests the self-consistency of the clean behavior in the large length-scale limit. New physics that the disorder may induce at finite scales is invisible to the Harris criterion.

Just as the Imry-Ma criterion, the Harris criterion (6) was originally derived for classical phase transitions. It takes the same form for zero-temperature quantum phase transitions because quenched disorder varies only in space but not in (imaginary) time. (The dimensionality d in Harris' inequality (6) is *not* replaced by $d + 1$ or $d + z$ in the quantum case.)

Harris' original criterion (6) which applies to uncorrelated spatial disorder has been generalized in several directions. For extended defects, i.e., disorder perfectly correlated in at least one space dimension, the inequality reads $d_\perp \nu > 2$ where $d_\perp$ is the number of dimensions in which there is randomness ($d_\perp = d - 1$ for line defects and $d_\perp = d - 2$ for plane defects). If the disorder features isotropic long-range correlations in space that decay as $|\mathbf{x} - \mathbf{x}'|^{-a}$, the Harris criterion is modified to $\min(d, a)\nu > 2$, making long-range correlated disorder with $a < d$ more relevant than uncorrelated disorder (68). Harris-like criteria can also be derived for disorder that varies in time or in space and time. For purely time-dependent disorder with short-range correlations, the resulting inequality reads $z\nu > 2$ where z is the dynamical critical exponent (69, 70). Recently, Vojta and Dickman derived a criterion for arbitrary spatio-temporal disorder in terms of its space-time correlation function (71). It contains the older results as special cases but also works for more complicated situations such as diffusive disorder degrees of freedom.

Another generalization of the Harris criterion is due to Luck (72) who considered the stability of critical points not just against random disorder but against a broader class of inhomogeneities whose fluctuations can be characterized by a wandering exponent ω. In terms of this exponent, the stability criterion reads $\omega < 1 - 1/(d\nu)$. The Harris-Luck criterion has been used, for example, for systems with quasiperiodic inhomogeneities.

Violations of the Harris criterion are sometimes reported in the literature, for example, for phase transitions on random Voronoi lattices (see Ref. (73) and references therein) or in certain dimerized spin models (74, 75). In the former case, they stem from hidden anticorrelations of the disorder variables caused by a topological constraint (73). The violations in the latter systems have been attributed to the fact that the disorder causes no (or extremely small) shifts of the local transition point.

Finally, we emphasize that the Harris criterion (6) tests the stability of the *clean* critical

Marginal case: the example of the 2d Ising universality class

The correlation length exponent $\nu = 1$ of the 2d Ising universality class is exactly marginal w.r.t. the Harris criterion, $d\nu = 2$. Is random-T_c disorder relevant or irrelevant? The critical behavior of a 2d disordered Ising magnet has been controversially discussed for a long time, but recent high-accuracy Monte Carlo simulations (82) provide strong evidence in favor of the strong-universality scenario (83, 84, 85) according to which the critical behavior is controlled by the clean Ising fixed point. Disorder is marginally *irrelevant* and gives rise to universal logarithmic corrections to scaling (see Ref. (82) and references therein). Interestingly, the same clean Ising behavior with logarithmic corrections also governs the critical point of the disordered N-color Ashkin-Teller model that emerges when the clean first-order transition is destroyed by disorder (82, 86, 87).

point and contains the correlation length exponent ν of the *clean* system. The separate question which value ν takes at the disordered critical point was addressed by Chayes et al. (76) who showed that the finite-size correlation length exponent in a disordered system must fulfill the same inequality $d\nu \geq 2$. However, there are unresolved questions about the relation between the finite-size correlation length exponent and the intrinsic one (77).

4. DISORDERED PHASE TRANSITIONS

So far, we have discussed the stability of clean phases and phase transitions against (weak) random-field and random-T_c disorder. We now turn to the ultimate fate of a transition in the presence of disorder. The focus will be on critical points because first-order phase transitions cannot exist in disordered system for $d \leq 2$, and comparatively little is known about disordered first-order (quantum) phase transitions in $d > 2$. Parts of this topic have been reviewed recently in Refs. (18, 19, 78, 79). We therefore only summarize the key concepts and emphasize the refined classification developed in Ref. (20).

4.1. Clean vs. finite-disorder vs. infinite-disorder critical points

Critical points in disordered systems can be categorized according to the behavior of the disorder strength under coarse graining (80). Three cases can be distinguished:

(i) If the Harris criterion is fulfilled, the disorder strength goes to zero under coarse graining, i.e., disorder is irrelevant in the renormalization group sense. The resulting critical behavior equals that of the clean transition, and macroscopic observables are self-averaging.

(ii) The second case comprises critical points at which the system remains inhomogeneous, and the (relative) strength of the disorder approaches a nonzero constant in the large length scale limit. These "finite-disorder" critical points generally show conventional power-law critical behavior, but the critical exponents differ from the corresponding clean ones. Macroscopic observables are not self-averaging at criticality; their distribution retains a finite width in the thermodynamic limit (81).

(iii) In the third case, the disorder strength (the relative magnitude of the inhomogeneities) goes to infinity in the limit of large length scales. The resulting infinite-disorder (or infinite-randomness) critical points usually show unconventional activated scaling behavior (88, 89) featuring an exponential relation between correlation length and time rather than the usual power-law relation.
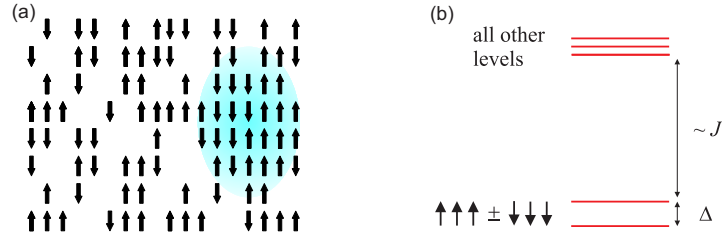
(a) Rare region in a diluted ferromagnet. The shaded region is impurity-free and thus behaves as a finite-size piece of the undiluted system. (b) Energy spectrum of a single rare region in a quantum Ising magnet. In the two low-energy states, all spins on the rare region are aligned. They are separated from all other states by a large gap of the order of the interaction energy J.

4.2. Rare regions and Griffiths singularities

Recent research has shown that many phase transitions in disordered systems are dominated by rare strong disorder fluctuations and the rare spatial regions on which they reside. Rare regions cause off-critical singularities in the free energy called the Griffiths singularities (90).

The importance of rare regions can be discussed using the example of a diluted ferromagnet shown in Fig. 3a. Due to statistical fluctuations of the vacancy positions, a macroscopic sample contains a small but nonzero concentration of large vacancy-free regions. If the system as a whole is close to the transition but still on the paramagnetic side, such regions can be locally ferromagnetic, i.e., their spins lock together and align parallel.

To decide whether or not rare regions play a significant role, one must estimate their total contribution to thermodynamic quantities. The probability for finding a large vacancy-free region of size L_{RR} is exponentially small in its volume $V_{RR} \sim L_{RR}^d$ and in the vacancy concentration c. Up to pre-exponential factors it reads $w(V_{RR}) \sim \exp(-cV_{RR})$.[7] Consequently, rare regions are important only if the contribution each one makes increases exponentially with its volume. At generic classical transitions, this is not the case. Each locally ordered region in a diluted ferromagnet, for example, acts as a superspin whose moment is proportional to the volume V_{RR}. The susceptibility of the rare region thus behaves as $\chi(V_{RR}) \sim V_{RR}^2/T$. As this power-law increase cannot overcome the exponential decrease of the rare region probability with V_{RR}, large rare regions do not make significant contributions. Thermodynamic Griffiths singularities in generic classical systems are thus weak essential singularities that are likely unobservable in experiments (92, 93, 94).

Quantum systems at zero temperature can have stronger Griffiths singularities. Consider, for instance, the energy spectrum of a rare region in a diluted Ising magnet in a transverse magnetic field, as sketched in Fig. 3b (95). The two low-lying states are the symmetric and antisymmetric combinations of the perfectly aligned "superspin" states. They are separated by an exponentially small gap $\Delta \sim \exp(-aV_{RR})$, leading to to an exponential increase of the rare region magnetic susceptibility with V_{RR}. The Griffiths singularities are therefore much stronger and of power-law from (96, 97). Power-law Griffiths singularities

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[7]This estimate holds for uncorrelated disorder. If the disorder features long-range correlations that decay as $|\mathbf{x} - \mathbf{x}'|^{-a}$ with $a < d$, rare regions are much more likely to occur. Their probability is enhanced and reads $w(V_{RR}) \sim \exp(-cV_{RR}^{a/d})$ (91).

can also appear at classical transitions in systems with extended defects, i.e., if the disorder is perfectly correlated in at least one dimension (98).

Even stronger rare region effects occur if the dynamics of an individual rare region can freeze independently of the bulk system. At $T \neq 0$, this can happen if the disorder is correlated in at least two dimensions (99, 100). At quantum phase transitions it can also be caused by the coupling of the order parameter to a dissipative bath (101, 102, 103).

4.3. Classification of disordered critical points

Vojta and Hoyos (20) recently showed that there is a deep connection between the Harris criterion and rare region physics. This allowed them to combine the two ways of categorizing critical points introduced in Secs. 4.1 and 4.2, leading to an improved classification scheme for classical, quantum, and nonequilibrium critical points under the influence of random-T_c disorder that extends earlier work (18, 104). The three main classes are determined by the relation of the effective rare region dimensionality d_{RR} with the lower critical dimension d_c^- of the phase transition.[8] For quantum phase transitions, the effective dimensionality includes the imaginary time direction as one of the dimensions.

Class A: If the dimensionality of the rare regions is below the lower critical dimension, $d_{RR} < d_c^-$, individual rare regions cannot order by themselves. Their contribution to thermodynamic observables grows at most as a power of their volume which cannot overcome the exponential decrease of their probability $w(V_{RR})$. Rare regions therefore make a negligible contribution to the critical thermodynamics. Transitions in this class include generic thermal (classical) transitions with uncorrelated disorder ($d_{RR} = 0$). Some quantum phase transitions also belong to this class, such as the transition in the diluted bilayer antiferromagnet (105, 106, 107) or the superfluid-Mott glass transition (108, 109, 110). Here, $d_{RR} = 1$ because the disorder is perfectly correlated in imaginary time but $d_c^- = 2$ because of the Mermin-Wagner theorem (111). Class A contains two subclasses depending on the Harris criterion. In subclass A1, the disorder strength asymptotically scales to zero, leading to clean critical behavior. Subclass A2 contains finite-disorder critical points with conventional power-law scaling but exponents that differ from the clean ones. For quantum phase transitions, this implies that the dynamical critical exponent z remains finite.

Class B: In this class, the rare regions are right at the lower critical dimension, $d_{RR} = d_c^-$, but still cannot undergo the transition by themselves. The rare region contribution to thermodynamic quantities now increases exponentially with their volume, compensating for the exponential decrease of the rare region probability. This leads to strong power-law Griffiths singularities controlled by a non-universal Griffiths dynamical exponent z'.

Class B is also divided into two subclasses according to the Harris criterion. In subclass B1, the disorder strength scales to zero for large length scales. Power-law Griffiths singularities coexist with clean critical behavior, and the Griffiths dynamical exponent z' does not diverge but approaches the clean z. Such behavior was recently found at a nonequilibrium transition (112); it may also explain the stability of the Belitz-Kirkpatrick critical behavior in weakly disordered metallic ferromagnets (see Refs. (32, 113)). In subclass B2, the disorder strength diverges for large length scales, giving rise to infinite-disorder criticality with activated scaling (z is formally infinite). Examples include thermal transitions in systems

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[8]The lower critical dimension d_c^- is the dimension below which the ordered phase is destroyed by fluctuations.

Table 1 Classification of critical points in the presence of random-T_c disorder according to the Harris criterion $d\nu > 2$ and the relation between the rare region dimensionality d_{RR} and the lower critical dimension d_c^- (after Ref. (20)).

Class	RR dimension	Subclass	Harris criterion	Griffiths Singularities	Critical behavior
A	$d_{\mathrm{RR}} < d_c^-$	A1	$d\nu > 2$	weak exponential	clean
		A2	$d\nu < 2$	weak exponential	convent. finite disorder
B	$d_{\mathrm{RR}} = d_c^-$	B1	$d\nu > 2$	power law, z' remains finite	clean
		B2	$d\nu < 2$	power law, z' diverges	infinite disorder
C	$d_{\mathrm{RR}} > d_c^-$			rare regions freeze	smeared transition

with extended defects such as the McCoy-Wu model (98) ($d_{RR} = d_c^- = 1$) or Heisenberg magnets with plane defects (114, 115) ($d_{RR} = d_c^- = 2$). Subclass B2 also contains the quantum phase transitions in the random transverse-field Ising model (88, 89, 97, 80), metallic Heisenberg magnets (116, 117), and superconducting nanowires (116, 118, 119). Disordered absorbing-state transitions also belong to this subclass (120, 121, 122, 123, 124).

Class C: In class C with $d_{RR} > d_c^-$, individual rare regions can undergo the phase transition independently of the bulk system. The global phase transition is smeared because a nonzero global order parameter arises as superposition of many independent rare regions, each with its own transition point. As the spatial correlation length does not diverge in this scenario, the Harris criterion does not play a qualitative role. Smeared classical phase transitions have been discovered in randomly layered Ising magnets (99, 100) ($d_{RR} = 2$ and $d_c^- = 1$). Smeared quantum phase transition include those in metallic Ising magnets (102) in the dissipative transverse-field Ising model (103, 125). Absorbing state transitions with extended defects also fall into this class (126, 127).

This classification, summarized in Table 1, applies to continuous transitions with random-T_c (random mass) disorder and sufficiently short-ranged interactions. It assumes that the coupling between the rare regions can be neglected. Long-range interactions such as the RKKY interaction in metals may thus lead to modifications (128).

5. QUANTUM GRIFFITHS PHASES

In broad terms, a Griffiths phase is a region in the phase diagram of a disordered system in which the randomness causes finite-size spatial regions to be locally in the wrong phase. Griffiths phases can appear on both sides of a phase transition; this is illustrated for a ferromagnet in Fig. 4. In the paramagnetic (disordered) Griffiths phase, locally ordered rare regions are embedded in the paramagnetic bulk system. In the ferromagnetic (ordered) Griffiths phase, in contrast, the bulk system displays long-range order. The rare regions are not simply holes in the magnetic order because these holes do not have an associated degree of freedom. Instead, they are locally ordered clusters inside the holes.[9]

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[9]Are Griffiths phases distinct phases or just parameter regions within a phase? From a symmetry perspective, a Griffiths phase is indistinguishable from its parent. A paramagnetic Griffiths phase, for instance, has the same symmetries as a conventional paramagnet. However, other qualitative features differ, for example, Griffiths phases are gapless even if their parent phases are gapped.
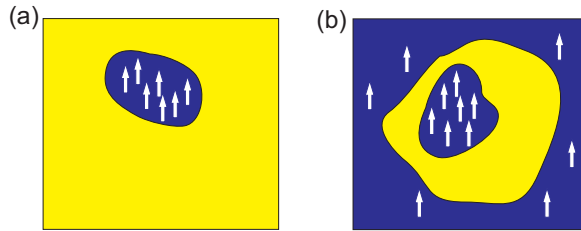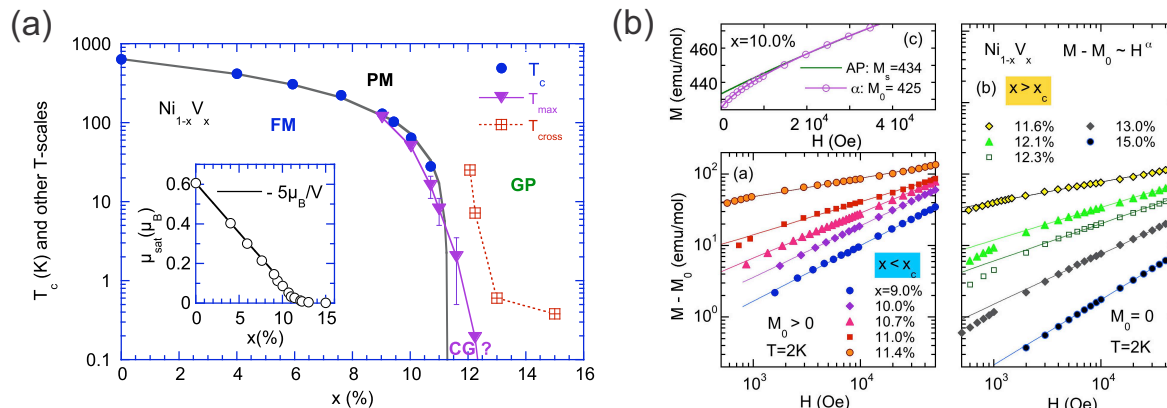
Figure 4

Schematic of the rare regions responsible for the Griffiths phases associated with a ferromagnetic transition. Left: The paramagnetic Griffiths phase is due to rare locally ferromagnetic regions embedded in the paramagnetic bulk. Right: The ferromagnetic Griffiths phase is caused by locally ferromagnetic regions located inside paramagnetic "holes" in the bulk ferromagnet.

Griffiths phases generically appear close to all phase transitions in disordered many-body systems, be they thermal, quantum or non-equilibrium transitions. Here, we focus on quantum Griffiths phases, i.e., Griffiths phases that are associated with zero-temperature quantum critical points. The phenomenology of a quantum Griffiths phase crucially depends on which class of the classification in Sec. 4.3 the critical point belongs to.

For transitions in class A, the rare region density of states decays exponentially at small energies. Consequently, rare region contributions to thermodynamic quantities are exponentially suppressed. A prototypical example of a quantum Griffiths phase in this class is the Mott glass phase emerging in systems of disordered bosons with particle-hole symmetry (108, 110, 129, 130). The Mott glass consists of superfluid "puddles" embedded in an insulating host; it is an incompressible insulator just like the conventional Mott insulator. Whereas the Mott insulator is gapped, the Mott glass is gapless, but with an exponentially small density of states at low energies. Rare regions contributions cause the compressibility to vanish as a stretched exponential with temperature, $\kappa \sim \exp(-\text{const}/T^{1/2})$, i.e., much slower than the conventional behavior $\kappa \sim \exp(-\text{const}/T)$. This behavior is an example of an essential Griffiths singularity, as is typical for class A.

Quantum Griffiths phases in class B feature much stronger Griffiths singularities because the combination of the exponentially decreasing rare region probability and the exponential dependence of their energy gap (or inverse characteristic time) on their size leads to a power-law density of states $g(\epsilon) \sim \epsilon^{d/z'-1}$ that is controlled by the nonuniversal Griffiths dynamic exponent z'. The resulting power-law quantum Griffiths singularities were first found in random transverse-field Ising models (88, 89, 97, 80). Later, they were also predicted to occur in disordered itinerant Heisenberg magnets (116, 117) and near the pairbreaking superconductor-metal quantum phase transition (116, 118, 119).

Perhaps the most convincing experimental example of a (class B) quantum Griffiths phase has been found in the random alloy $Ni_{1-x}V_x$. Nickel is a ferromagnet with a Curie temperature of 627 K. Alloying with vanadium quickly suppresses the ferromagnetism leading to a quantum phase transition to paramagnetism at a critical vanadium concentration x_c between 11% and 12% (see Fig. 5a). Ubaid-Kassis et al. (131) identified a Griffiths phase on the paramagnetic side of the quantum phase transition ($x > x_c$) that shows the predicted power-law behaviors of the susceptibility, $\chi(T) \sim T^{d/z'-1}$, and the magnetization-field curves, $M(H) \sim H^{d/z'}$ (see Fig. 5b). More recently, Wang et al. (132) discovered a corresponding Griffiths phase inside the ferromagnetic phase.

Figure 5

(a) Phase diagram of $Ni_{1-x}V_x$. FM and PM denote the ferromagnetic and paramagnetic phases. The (disordered) Griffiths phase (GP) emerges at low temperatures and x slightly above x_c. At the lowest temperatures and close to x_c, there may be a cluster glass (CG) phase (from Ref. (131). (b) Low-temperature magnetization-field curves of $Ni_{1-x}V_x$ on both sides of the quantum phase transition. For $x > x_c$, they can be fitted with $M \sim H^\alpha$ with $\alpha = d/z'$ the Griffiths exponent. For $x < x_c$ they behaves as $M - M_0 \sim H^\alpha$ where M_0 is the spontaneous magnetization (from Ref. (132)).

Several other examples of magnetic quantum Griffiths phases in metallic systems have been found in recent years (see Refs. (19, 79) and references therein). In 2015, Xing et al. (133) reported Griffiths singularities near the superconductor-metal transition in Ga thin films. Moreover, the elusive "sliding" Griffiths phase predicted to occur in layered superfluids (134, 135) may have been observed in a system of ultracold atoms.

For quantum phase transitions in class C of the classification, the quantum Griffiths phase is replaced by a tail of the conventional long-range ordered phase because the dynamics of sufficiently large rare regions freezes at zero temperature (102). The question whether or not Griffiths singularities can be observed at elevated temperatures has been discussed controversially in the literature (136, 137, 101, 138). Evidence for a smeared quantum phase transition was found in $Sr_{1-x}Ca_xRuO_3$ thin films (139). Pure $SrRuO_3$ is ferromagnetic whereas $CaRuO_3$ is paramagnetic. The dependence of the critical temperature as well as the magnetization on the Ca concentration x agree well with the smeared phase transition scenario for itinerant Ising magnets, adapted to the case of composition-tuning (140, 141).

6. CONCLUSIONS AND OUTLOOK

In conclusion, we have reviewed the stability of phases and phase transitions in many-body systems against impurities, defects and other types of quenched disorder. We have focused on the physics on large length scales where even weak disorder can lead to qualitative changes of phases and transitions.

A general theme has emerged from this discussion: When disorder is destroying a long-range ordered phase or a clean phase transition, exotic new states of matter are likely to appear that are interesting in there own rights and do not have clean counterparts. In the following, we summarize the main points and list a few open issues.

SUMMARY POINTS

1. Random-field disorder prevents spontaneous symmetry breaking in $d \leq 2$ for discrete order parameter symmetry and in $d \leq 4$ for continuous symmetry.
2. Random-field disorder arises naturally for order parameters that break real-space symmetries.
3. If long-range order is destroyed by random fields, exotic "glassy" phases such as the Bragg, spin-density-wave, and pair-density-wave glasses can emerge.
4. Weak random-T_c disorder does not prevent spontaneous symmetry breaking, but it can destroy first-order phase transitions and destabilize clean critical points.
5. Critical points in disordered systems feature unconventional scaling scenarios that can be classified according to the rare-region dimensionality and the Harris criterion.
6. Exotic Griffiths phases emerge near disordered critical points, including the Mott and Bose glasses, the itinerant ferromagnetic quantum Griffiths phase, and the sliding phase in layered superfluids.

FUTURE ISSUES

1. While the thermodynamics of many of these exotic phenomena is well understood, much less is known about the real-time dynamics and transport properties.
2. Theory cannot yet explain the transport properties near disordered quantum phase transitions in metallic systems.
3. How does disorder interact with phases characterized by several intertwined orders? Does it promote or hinder the formation of vestigial orders?
4. What are the effects of disorder on phases and phase transitions that do not follow Landau's paradigm?

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