Mean-field model for electron-glass dynamics

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We study a microscopic mean-field model for the dynamics of the electron glass near a local equilibrium state. Phonon-induced tunneling processes are responsible for generating transitions between localized electronic sites, which eventually lead to the thermalization of the system. We find that the decay of an excited state to a locally stable state is far from being exponential in time and does not have a characteristic time scale. Working in a mean-field approximation, we write rate equations for the average occupation numbers $\langle n_i \rangle$ and describe the return to the locally stable state by using the eigenvalues of a rate matrix A describing the linearized time evolution of the occupation numbers. By analyzing the probability distribution $P(\lambda)$ of the eigenvalues of A, we find that, under certain physically reasonable assumptions, it takes the form $P(\lambda) \sim \frac{1}{|\lambda|}$, leading naturally to a logarithmic decay in time. While our derivation of the matrix A is specific for the chosen model, we expect that other glassy systems, with different microscopic characteristics, will be described by random rate matrices belonging to the same universality class of A. Namely, the rate matrix has elements with a very broad distribution, as in the case of exponentials of a variable with nearly uniform distribution.

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I. INTRODUCTION

Experiments conducted on thin films of amorphous or crystalline semiconductors such as indium oxide or silicon show that when driven out of equilibrium (for example, by shining light on the system or by changing a gate voltage), the system exhibits slow relaxations, observable on the scale of minutes or hours.[1,2] In many cases, a logarithmic or weak power-law time dependence of the measured quantity (such as conductance and capacitance) is observed over many decades of time.[3–5] A common feature of the experimental systems is that they are highly disordered, so that most electronic states are localized. If the carrier concentration is high enough,[6] the (unscreened) Coulomb interactions may play an important role.[7] This system is usually referred to as the electron glass since it exhibits many features characteristic of glassy systems: memory effects[8] (the conductance depends on the previous perturbations applied to the system) and aging[9] (the duration of time the perturbation is applied affects the relaxation process). Similar effects have been observed in granular Al,[10,11] showing that the underlying principles may be more general.

In this paper, we study a mean-field model for the dynamics of the system. A variety of systems in nature can be described near a locally stable state by a matrix equation of the type

$$\frac{d\vec{\delta n}}{dt} = A \cdot \vec{\delta n}, \tag{1}$$

where the component $\delta n_i = n_i - f_i$ is the deviation of the average occupation of the ith site, n_i, from its value f_i at the locally stable point. The local stability of the point implies that the matrix A must have only nonpositive eigenvalues and, for large systems, their distribution will determine the average time dependence of the return to the locally stable point, after the system was slightly pushed away from it. We will show how this picture emerges from the mean-field approximation of the electron glass. It must be emphasized that our approach is different from the usual theoretical explanations of aging phenomena in glasses, in which the system explores the energy landscape, and slow relaxations are a result of the existence of many metastable states. In our model, the system is found in the vicinity of one locally stable point at all times (we do not use the term metastable to stress this difference). This assumes that the initial perturbation is small enough (and so is the temperature), such that the system does not reach other (lower) minima but remains in the same region of phase space. Slow relaxations are due to isolated localized states that, statistically, happen to have a long lifetime. It should be emphasized that although the interactions lead to the nontrivial Coulomb gap[12] in the equilibrium state, the slow dynamics will also occur without interactions.

If it is given that the distribution of eigenvalues diverges at small (negative) eigenvalues and is of the form $P(\lambda) \sim \frac{1}{|\lambda|}$ (as happens in our model), it is straightforward to see that logarithmic relaxation in time, in an appropriate time window, is obtained (assuming that the eigenvectors are excited with uniform probability). In this work, we show that starting from a realistic microscopic model for the electron-glass system, the described situation indeed occurs, and we argue that it is plausible that other physical systems will also show similar results.

The structure of the paper is as follows. The model is defined in Sec. II A. In Sec. II B, we review the application of the mean-field approximation to the peculiar equilibrium properties of the system, manifesting the Coulomb gap. In Sec. II C, we briefly discuss the mean-field steady-state solution in the presence of an external field, leading to the Miller–Abrahams model.[13]

In a similar fashion, in Sec. III, we suggest to study the dynamics of the system by writing a set of ordinary differential equations, given by Eqs. (2) and (6), giving the time evolution of the occupancies of the localized states. This is

already an approximation neglecting interference or quantum fluctuation effects. In Sec. III A, we study the dynamics of the electron glass, starting from an out-of-equilibrium state. By linearizing Eq. (2), we obtain the time-evolution equations of the occupations and obtain Eq. (1), with the random matrix A belonging to a different class from the Gaussian random matrix ensembles. The statistics of the eigenvalues is studied numerically (see Fig. 4). In Sec. III B, we analytically study a simplifying limit. Both lead to a distribution of eigenvalues $P(\lambda)$ diverging at low values (down to a cutoff), leading to slow relaxations of the physical observables, as seen experimentally. This behavior might be characteristic of glassy systems. Finally, in Sec. III D, we discuss the relation between the relaxation of the occupation numbers and the conductance.

II. MEAN-FIELD MODEL FOR ELECTRON GLASS

In this section, we discuss a specific microscopic model for the dynamics of the electron glass. We will show that it leads to a rate equation of the type of Eq. (1) and explicitly find the matrix elements.

A. Definition of the model

We study a system of N localized states and $M<N$ electrons with a coupling between the electrons and a phonon reservoir.[14] Since the states are localized, the electrons will interact via an unscreened Coulomb potential. In the absence of electron-electron interactions, the localized states have different energies ϵ_i due to the disorder. Our model also contains structural disorder: the positions of the sites are assumed to be random. Although localized states are orthogonal, their tails overlap and, therefore, phonons may induce transitions between them. The generic coupling between electrons and phonons is given by the form $\Sigma_q M_q c_i^\dagger c_j (b_q^\dagger + b_{-q})$, where $c_i^\dagger, c_j$ are electron creation and annihilation operators at local sites i, j and b_q annihilates a phonon. M_q is a coefficient accounting for the strength of the electron-phonon coupling.

Let us denote the energy difference of the electronic system before and after the tunneling event by ΔE, containing the interaction effects. For weak electron-phonon coupling ($|M_q|^2 \nu \ll \Delta E$, where ν is the phonon density of states), the transition rate γ_{ij} of an electron from site i with energy E_i to site j with energy $E_j<E_i$ a distance r_{ij} away can be calculated, treating the coupling as a perturbation. This yields, up to polynomial corrections,[12]

$$\gamma_{ij} \sim |M_q|^2 \nu f_i (1-f_j) e^{-r_{ij}/\xi} [1 + N(\Delta E)], \tag{2}$$

where f_i is the Fermi–Dirac distribution. For upward transitions ($E_j>E_i$), the square brackets are replaced by $N(\Delta E)$. These rates may be renormalized due to polaron-type orthogonality effects,[15] which may significantly increase the time scales.

We will be interested in the dynamics of the system when it is out of equilibrium, namely, in the time dependence of the occupation numbers and the conductance after an initial excitation. Nevertheless, we first show how the nontrivial equilibrium properties are obtained from the mean-field picture.

B. Equilibrium properties near a locally stable point

In an approximation similar to those used in spin glass theory,[16] we define $f_i=\langle n_i \rangle$, where n_i is the site occupation (which takes the values zero or one) and $\langle\ \rangle$ denotes ensemble averaging over the thermal phonon bath. This is a mean-field-type approximation and may be used regardless of the existence of interactions in the system. The approximation improves as the number of interacting sites increases.[17] The long-range nature of the interaction means that the energy of a site will be determined by many of its neighbors and gives intuitive justification of the use of mean-field theory. Nevertheless, we should emphasize that this is an uncontrolled approximation, and the limits of its validity should be checked.

Let us first discuss the thermal equilibrium state near the locally stable point. The site occupation must follow the Fermi–Dirac distribution and, therefore,

$$f_i(E_i) = \frac{1}{1 + e^{(E_i-\mu)/T}}, \tag{3}$$

where μ is the chemical potential and the Boltzmann constant is set to be 1.

In the mean-field approximation, we can calculate the average potential energy of site i,

$$E_i = \epsilon_i + \sum_{j \neq i} \frac{e^2 f_j}{r_{ij}}. \tag{4}$$

By combining Eqs. (3) and (4), one obtains a self-consistent equation for the energies. It is common to use an unbiased disorder distribution and add a background charge ν to each site.[18] In the mean-field picture, this will lead to the equation

$$E_i = \epsilon_i + \sum_j \left(\frac{1}{1 + e^{(E_j-\mu)/T}} - \nu \right) \frac{e^2}{r_{ij}}. \tag{5}$$

For half filling, $\mu=0$ and $\nu=0.5$.

Rigorously, one cannot call any solution an equilibrium distribution since the equilibrium distribution is a Boltzmann average over *all* configurations, not only those near the locally stable point. The solution may be viewed as a "local equilibrium." We will see that the physical picture obtained is quite plausible. At low temperatures,[19] the probability distribution of the energies will contain a soft gap at the Fermi energy, known as the Coulomb gap,[12,18,20–25] as we shall discuss now.

Equation (5) can be numerically solved by starting with a random set of energies and evolving them iteratively within the mean-field model. This was done following Ref. 18 by solving the equations for many random instances and averaging over them. In this way, a histogram of the on-site energies is obtained. When normalized correctly, it gives the

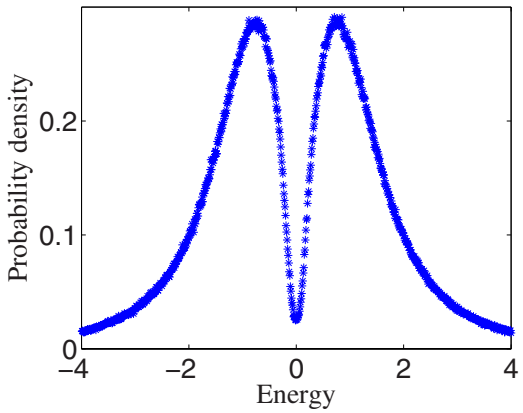

FIG. 1. (Color online) Histogram of the site energies in two dimensions, for $N=10\,000$ and $\mu=0$ (half filling), obtained by solving the self-consistent Eq. (5). The sites were uniformly distributed in a square with $\frac{e^2}{r_{nn}T}=20$, where r_{nn} denotes the average nearest-neighbor distance and T is the temperature. The energies ϵ_i were uniformly distributed in the interval $[-\frac{W}{2},\frac{W}{2}]$ with $W=1$. The y axis denotes the probability density of the energies E_i. The graph is the average over 300 instances. Notice the finite value of the density at the minimum due to the finite temperature.

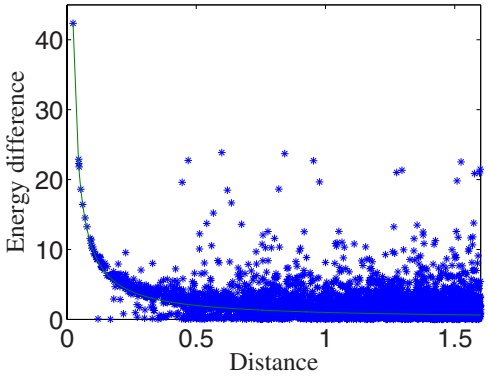

FIG. 2. (Color online) Demonstration of correlations between the distance and energy difference of two sites chosen at random. The parameters were as for Fig. 1 with $N=1000$. The graph shows the energy difference of all the pairs of sites as a function of their distance. The parameterless fit clearly shows that at small distances, $\delta E \sim \frac{e^2}{r}$.

single-particle density of states (DOS) as a function of energy. The results for two dimensions, yielding the Coulomb gap, are given in Fig. 1. Notice that the obtained DOS contains a linear gap near the Fermi energy, which is in accordance with other works.[20,24]

It should be mentioned that the energies of the sites are not independently distributed: in order for the sum of Eq. (5) not to diverge, there must exist spatial correlations in the energy distribution among the sites. The fact that correlations exist should come as no surprise: for example, let us look at two sites which are close to each other. At low temperatures, we expect the solution of the equations to have the occupancy at one of the sites close to unity and small for the other site. Since the influence of the other sites on the energy difference of the two nearby sites will be small (only dipole corrections), the energy difference between the two sites will be $\sim\frac{e^2}{r}$. This is demonstrated in Fig. 2.

C. Response to an external field: Steady-state solution

When a small electric field is applied, there are corrections to the average occupations and also to the average energies. It can be shown that the problem of finding the steady-state solution corresponds to that of solving the steady state of a resistance network, using Kirchoff's laws.[13] The solution, when neglecting interactions, gives the well-known Mott variable range hopping,[26] which was experimentally observed in many cases.[27] This calculation can be performed through the mean field: the steady-state solutions are obtained from time-dependent equations that are essentially the mean-field equations. In the following, we propose to use the same ideas to discuss the dynamics of the system out of equilibrium.

III. DYNAMICS

Let us pose the following question: How will the occupation numbers or conductance depend on time when the system is pushed slightly out of the locally stable point? Having seen that the mean-field approximation yields the correct density of states as well as the out-of-equilibrium steady-state solution in the presence of an external field, we propose to use the same approximation to describe the dynamics of the systems when prepared out of local equilibrium.

Experimentally, the form of the relaxation depends on the details and mechanism of the excitation. For simplicity, let us assume that the initial perturbation takes the form of a random addition δn_i to the state occupations, with $\Sigma_i \delta n_i=0$, reflecting particle number conservation.

Assuming that the initial change in the occupations is small, we can still use Eq. (2) for the tunneling rates, with the average occupations at the locally stable point f_j substituted by the occupation numbers slightly out of equilibrium n_j (which can take any value between 0 and 1). The energies at each instance are related to the out-of-equilibrium occupations by Eq. (4), upon replacing f_j by n_j, and we can write the time evolution of the average occupation as

$$\frac{dn_i}{dt}=\sum_{j\neq i}\gamma_{ji}-\gamma_{ij}. \tag{6}$$

This defines the problem completely. At the locally stable point itself, the right-hand side of Eq. (6) must vanish. Therefore, not too far from the point, we can take the first (linear) order in the quantities δn_i, the deviations from the stable point. The linearized equation then takes the form of Eq. (1), where $\vec{\delta n}$ is a vector of the deviations of the occupation numbers from their local equilibrium values. The N eigenvalues and eigenvectors of the matrix A will determine the decay rates of the system.

We would like to stress that the matrix A can be calculated for the cases of interest by linearizing the equations of motion *near the locally stable point*. This strategy is completely general and will be valid for any system, which can

be described by equations of motion and has a locally stable point (such would be the case for most classical systems and many quantum systems in a mean-field approximation). The dynamics of the system when pushed slightly away from the fixed point will be characterized by the eigenvalues of the rate matrix. In the common case where disorder plays a role, the dynamics will depend on the distribution of eigenvalues of the matrix: this is, in fact, a problem of random matrix theory,[28] where the eigenvalue distribution is responsible for the *dynamics* of the system. An extremely relevant property of the electron-glass case, as we shall demonstrate in Sec. III A, is that the entries of the random matrix are *exponentials* of the broadly distributed parameters (energy and distance). Another important feature of these matrices is that the sum of every column vanishes. These properties make this matrix belong to a different class from the Gaussian ensembles usually treated in random matrix theory and will play an important role in the dynamics, leading to slow relaxations. A similar class of matrices was previously studied by Mezard *et al.*[29]

In the following section, we will derive the form of the matrix A for the particular case of localized states coupled through phonons. We will find that the probability distribution is divergent for small eigenvalues and suggest what the minimal properties leading to such a distribution are. The implications of this distribution on the time-dependent relaxation of the occupation numbers and conductance will then be discussed.

A. Application to the electron-glass model

Starting from Eq. (2), a calculation of the elements of matrix A in Eq. (6) shows that

$$A_{ii} = \sum_{j\neq i} -\frac{\gamma_{ij}^0}{n_j^0(1-n_j^0)}, \tag{7}$$

where γ_{ij}^0 are the local equilibrium rates, given by Eq. (2), and n_j^0 are the occupation numbers in equilibrium. For $i \neq j$,

$$A_{ij} = \gamma_{ij}^0 \frac{1}{n_j^0(1-n_j^0)} - \sum_{k\neq j,i} \frac{e^2\gamma_{ik}^0}{T}\left(\frac{1}{r_{ij}} - \frac{1}{r_{jk}}\right). \tag{8}$$

Notice that the matrix is not symmetric due to the $n_j^0(1-n_j^0)$ term. The sum of each column of the matrix vanishes, guaranteeing particle number conservation.

A priori one would expect that at low enough temperatures $T \ll \frac{e^2}{r_{nn}}$, we could neglect the first term in the equation for the regime of interest. However, at low temperatures, the occupations of the sites tend to 0 or 1 exponentially, implying that the $\frac{1}{n_j(1-n_j)}$ term explodes much faster than the $\frac{1}{T}$ part in the second term. Viewed in a different way, if one looks at the expression of the mean-field rates $\gamma_{ij} \sim n_i(1-n_j)[1+N(\Delta E)]$, one sees that if two states are close in energy, then the first term in the matrix element [Eq. (8)] $\sim \frac{n_i}{n_j} N(\Delta E) \sim \frac{T}{\Delta E}$. Therefore, there is good coupling between *any* two states close in energy (and distance), not only those ones close to the Fermi level, as is the case for the second term. Therefore, the "phase space" is much larger for the first term,

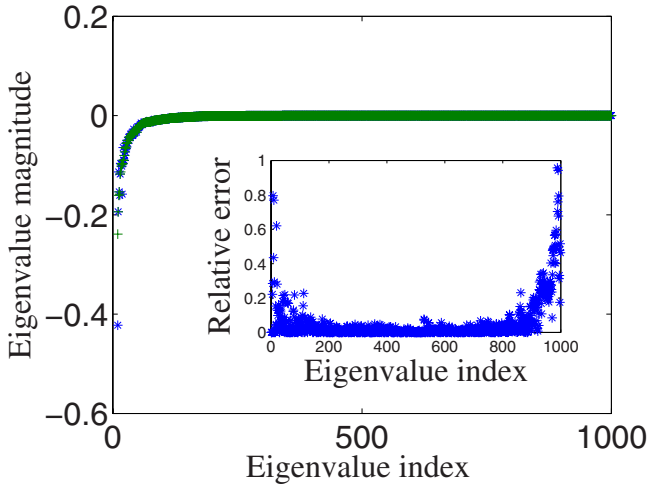

FIG. 3. (Color online) Comparison of the eigenvalue magnitudes of the full linearized matrix (stars) and the ones obtained after neglecting the "Coulomb" term (crosses), the second one in Eq. (8). The difference is mostly seen for the large magnitude eigenvalues, which are not relevant for the behavior at long time scales. $N=1000$, $\frac{\xi}{r_{nn}}=0.1$, and $\frac{e^2}{r_{nn}T}=10$. The energies ϵ_i were uniformly distributed in the interval $[-\frac{W}{2},\frac{W}{2}]$ with $\frac{W}{T}=10$. The inset shows the relative error in replacing the full matrix with the approximated one, which is defined as the difference between the approximation and exact diagonalization divided by the exact value.

and the second one can be neglected. We have numerically calculated the eigenvalue distribution for some specific system parameters and, indeed, it was verified that the second part has a small influence on most eigenvalues (see Fig. 3).

An important property follows: the off-diagonal elements are positive. Together with the property that the sum of every column vanishes, the stability of the mean-field solution is guaranteed: the real parts of all the eigenvalues are negative (or zero), characterizing decay.[30] In fact, this is a necessary and sufficient condition for the stability of the point.

Let us consider the distribution of the eigenvalues. Figure 4 shows the reciprocal of the distribution of the real part of the eigenvalues of the matrix A, as numerically obtained. We first found a mean-field solution by iterating the equations (see, for example, Ref. 18), then used Eq. (8) to construct the relevant matrix. The eigenvalues of this matrix were numerically found. Notice that the localization length ξ influences the dynamics, although it has no effect on the equilibrium properties, at least as long as the localized states are spatially well separated.

In Sec. III B, we will analyze a simplifying limit, when the rather complicated dependence on energy can be neglected, and consider only the exponential dependence of the tunneling rate on length. Both limits give approximately a $\frac{1}{|\lambda|}$ distribution (up to logarithmic corrections), reminiscent of $\frac{1}{f}$ noise.[31] This suggests that the result may be more general and not dependent on the details of the specific model. Note that the interactions affect the mean-field solution (and the Coulomb gap), but the calculation shows that the slow dynamics will exist also without them.

Let us now discuss the consequences of this distribution for the dynamics of the system. Having found that the distri-

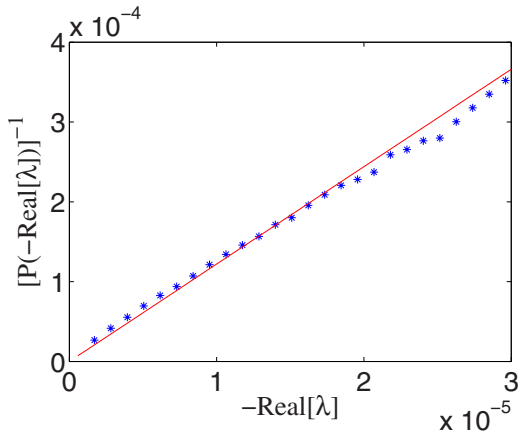

FIG. 4. (Color online) Reciprocal of the distribution of the real part of the eigenvalues, obtained numerically, averaged over 1000 realizations. The parameters were as for Fig. 3. The fit to the reciprocal distribution is linear.

bution is $\sim\frac{1}{|\lambda|}$ down to some minimal value, we conclude that if all eigenvectors (except the one with eigenvalue zero, not conserving particle number) are excited with equal probability, the time evolution of the deviation from the local stable point will be related to the Laplace transform of the eigenvalue distribution. This will give rise to logarithmic decays. Let us show this in more detail: if the eigenvectors are denoted by $\vec{n}_\lambda$, the time-dependent deviation from the locally stable point will be given by

$$\vec{\delta n} = \sum_\lambda c_\lambda \vec{n}_\lambda e^{-\lambda t}. \tag{9}$$

We shall assume that the eigenvectors are excited with roughly uniform probability. Going to the continuous limit, and utilizing the fact that the components of the eigenvectors themselves are also random variables, we obtain that the norm of $\vec{\delta n}$ should relax as

$$|\vec{\delta n}(t)| \sim \int_{\lambda_{min}}^{\lambda_{max}} \frac{e^{-\lambda t}}{\lambda} d\lambda \sim \gamma_E - \log[t\lambda_{min}], \tag{10}$$

for $\frac{1}{\lambda_{min}} < t < \frac{1}{\lambda_{max}}$, where γ_E is the Euler constant.

B. Dynamics of exponential models

Hitherto, we have discussed a specific model for glass dynamics in a system constructed from interacting electrons and phonons. The actual form of the rate matrix eigenvalue distribution did not strongly depend on the details of the matrix elements. In this section, we will show that there are few sufficient conditions on the random rate matrix A that will make the relaxation process long. Let us look at the dynamics which follow from a class of random matrices obeying the following properties.

(1) The sum of every column vanishes. This follows from particle number conservation.

(2) The entries of the matrices are distributed over a very broad range. This happens, for example, when they are exponentials of a more or less flat distribution.[32]

We expect that a variety of systems that exhibit a glassy behavior may be described by a random rate matrix belonging to this class. The matrix obtained for the electron-glass system indeed obeys these properties. The first property was explicitly shown in Sec. III A. To see the second, let us examine Eq. (8). If we neglect cases where the energies E_i, E_j, and $|E_i - E_j|$ are smaller than T, we can recast the equation into a more transparent form,

$$A_{ij} \sim e^{-r_{ij}/\xi} e^{-|E_i - E_j| - |E_i| + |E_j|/2T}. \tag{11}$$

Due to the exponential, the matrix entries are indeed broadly distributed.

We shall now discuss a specific class of matrices which can be analytically analyzed. As seen in Eq. (11), the matrix elements for the electron-glass system contain a factor $e^{-r_{ij}/\xi}$. If ξ is much smaller than the typical distance r_{nn}, it is plausible that this factor would be dominant in determining the eigenvalue distribution. This motivates us to discuss a simpler model of so-called distance matrices:[29] assume that we have N random points in a two dimensional space. Let us define a matrix $B_{ij} = e^{-r_{ij}/\xi}$, where r_{ij} is the distance between points i and j, and ξ some constant. Let us choose the diagonal elements of the matrix such that the sum of every column vanishes. Following the previous discussion of the dynamics, we are interested in the distribution of eigenvalues of such a matrix. A mapping of this problem to a field theory problem is given in Ref. 29, enabling one to look at a low-density approximation to the theory. Mezard *et al.*[29] calculated the resolvent $R = \frac{1}{N}\mathrm{Tr}\frac{1}{\lambda - H}$, the imaginary part of which yields the density of states, i.e., the distribution of eigenvalues. By using their formula (21) for the case of $f(r) = e^{-r/\xi}$, we obtain that the low-density expansion of the resolvent is

$$R(\lambda) = \frac{\rho}{2V} \int dxdy \left(\frac{1}{\lambda + 2e^{-r/\xi}} - \frac{1}{\lambda} \right), \tag{12}$$

where the integrals are performed over the whole volume. The second term gives rise to a delta function at the origin, which comes from the zero eigenvalue the matrix always possesses, and is of no particular interest since the eigenvector associated with this eigenvalue cannot be excited while preserving the particle number. The condition for the approximation to be valid is $\xi \ll r_{nn}$, as we shall show later in a more transparent way.

Since the density of states is given by $-\frac{\mathrm{Im}[R(\lambda)]}{\pi}$, we can use the fact that $\mathrm{Im}[\frac{1}{x + i\epsilon}] = -i\pi\delta(x)$ and obtain the DOS as

$$P(\lambda) = \frac{\rho}{2V} \int dxdy\, \delta(\lambda + 2e^{-r/\xi}). \tag{13}$$

Performing the integral in one dimension, for eigenvalues not too close to the minimal value $2e^{-L/\xi}$, leads to the result,

$$P(\lambda) = \frac{-N\xi}{L\lambda}, \tag{14}$$

with λ in the interval $[-2, -2e^{-L/\xi}]$. Repeating the calculation in two dimensions, again, for eigenvalues not too close to the minimal values, yields

$$P(\lambda) = \frac{\pi N \xi^2 \log\left(-\frac{\lambda}{2}\right)}{L^2 \lambda}. \tag{15}$$

We shall now give a transparent demonstration of these results. In the low-density limit, we can couple each site to its nearest neighbor, thus dividing the system into $\frac{N}{2}$ pairs. Neglecting the effect of all other sites, we obtain that the eigenvalues will be similar to those of an ensemble of 2×2 matrices of the form

$$M = \begin{pmatrix} -e^{-|x-y|/\xi} & e^{-|x-y|/\xi} \\ e^{-|x-y|/\xi} & -e^{-|x-y|/\xi} \end{pmatrix}. \tag{16}$$

Since the matrix elements of B decay on the scale of ξ, this would be a good approximation for $\xi \ll r_{nn}$. One eigenvalue of M is 0, and the other is $-2e^{-|x-y|/\xi}$. Therefore, half the eigenvalues of B will be the zero under this approximation, and the distribution of the other eigenvalues will be that of the random variable $-2e^{-r/\xi}$, where r is the nearest-neighbor distance. Notice that the zero eigenvalues correspond to the second term in Eq. (12).

The distribution of the nearest-neighbor distance can be calculated: looking at a typical site, let us calculate the probability that its nearest neighbor is at least distance r away. This is equivalent to asking that *all* of its neighbors are at least a distance r away, and since they are randomly distributed, we obtain that

$$\text{Prob}(r) = \left(1 - V_D \frac{r^D}{L^D}\right)^{N-1}, \tag{17}$$

where $V_1 = 2$ and $V_2 = \pi$.

For $r \ll L$, we can approximate

$$\text{Prob}(r) = e^{-V_D N (r^D/L^D)}. \tag{18}$$

We have assumed that the initial site is not too close to the boundaries. Since we are interested in the probability, the sites near the boundary will give a negligible correction to the above probability: the sites for which Eq. (18) fails are a distance of order r_{nn} or less from the boundary. Therefore, their fraction in the system is of order $\frac{r_{nn}}{L}$. For $N \gg 1$, this fraction is negligible.

The probability *distribution* can be calculated by differentiating with respect to r, leading to

$$P(r) = V_d D N \frac{r^{D-1}}{L^D} e^{-V_D N (r^D/L^D)}. \tag{19}$$

By construction, the probability distribution is exactly normalized. In one dimension, the eigenvalue distribution that follows is

$$P(\lambda) = -N \frac{e^{-2N\xi/L|\log(-\lambda/2)|}}{L\lambda} \xi \sim \frac{1}{\lambda^{1-\epsilon}}, \tag{20}$$

with $\epsilon = \frac{2N\xi}{L} \ll 1$, while for two dimensions,

$$P(\lambda) = \frac{\pi N \xi^2 \log\left(-\frac{\lambda}{2}\right) e^{-\pi(\xi^2/L^2)N \log^2(-\lambda/2)}}{L^2 \lambda}. \tag{21}$$

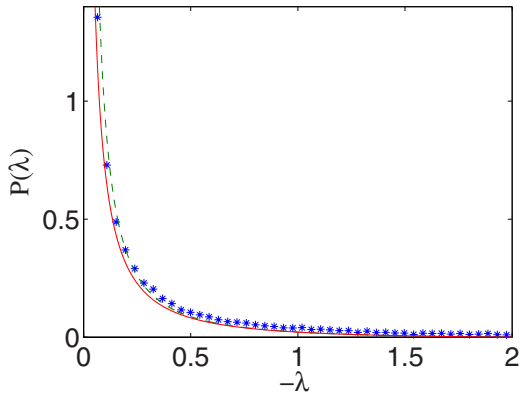

FIG. 5. (Color online) Distribution of the eigenvalues in two dimensions for a low density ($\frac{\xi}{r_{nn}} = 0.05$). $N = 1000$, and the results were obtained after averaging over 1000 instances. The points were uniformly distributed in the unit square, and the matrix element $A_{ij} = e^{-d_{ij}/\xi}$ with $\xi = 0.0016$. The dashed curve is a plot of Eq. (15), while the solid curve is a plot of Eq. (21).

Aside from the exponential term, Eqs. (20) and (21) coincide with the field theory [Eqs. (14) and (15)]. Notice that in the latter, there is a cutoff on the eigenvalue magnitude, while for Eqs. (14) and (15), the distribution is nonzero also for very small eigenvalues. Figure 5 shows the results of numerical simulations for the case of low-density distance matrices in two dimensions and a comparison to the theory.

C. Role of the interactions

The previous section demonstrated that a simplified model where the energy dependence of the matrix elements of A was neglected already leads to a $\frac{1}{\lambda}$ distribution of the real part of the eigenvalues. What, then, is the role played by the Coulomb interactions? For one, we know that they lead to the Coulomb gap, as demonstrated in Fig. 1. It is expected that this soft gap should influence the conductivity. A natural question one may ask is whether the interactions should affect the dynamics of the relaxation process and the time scales involved. Figure 6 shows the numerical result for the eigenvalue distribution at various values of the interaction strength. It is seen that $P(\lambda)$ follow the same $\frac{C}{\lambda}$ curve (with the same prefactor C), but the interactions shift the whole distribution to lower values: the lower and upper cutoff are shifted downward, so that the average eigenvalue is smaller due to the interactions. This can be explained as follows: as the interactions grow, the width of the site energy distribution grows as $\frac{e^2}{r_{nn}}$. Therefore, the typical matrix element of A will have a larger "penalty" for the energy mismatch between two random sites, driving the eigenvalues to be smaller yet. Nevertheless, it is quite remarkable that the same $\frac{C}{\lambda}$ distribution is followed, with the upper and lower cutoffs renormalized.

D. Relaxation of the conductance

In many cases, a logarithmic relaxation in time is observed for the conductance.[3,4,11] One should ask what is the

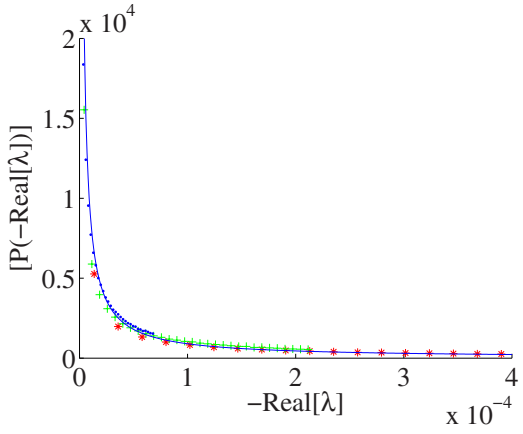

FIG. 6. (Color online) The effect of interactions on the distribution of the eigenvalues of A. The graph shows $P(\lambda)$ for various interaction strengths, each denoted by a different color and mark (dots are the case with strongest interaction, while stars are the weakest). The temperature, disorder, density, and localization length were as for Fig. 3, and $\frac{e^2}{r_{nn}T}$ was varied from 1 to 10. It is observed that $P(\lambda)$ follow the same $\sim\frac{1}{\lambda}$ dependence with the same prefactor, but as the interaction strength grows, the lower and upper cutoffs are pushed to lower eigenvalues.

relation between the relaxation of the conductivity and that of the occupations for the electron-glass model.

We now present an intuitive argument motivating the speculation that the time relaxation of the conductance should be similar to that of the occupations. The essence of the argument is the claim that at low temperatures, any perturbation of the equilibrium configuration will lead to enhanced conductance:[33] if this is true, it is reasonable that as the typical deviation of the occupation number relaxes, so does the enhanced conductivity, until it reaches its equilibrium value. For small enough deviations, the two will be proportional to each other, as one can always take the lowest order term in the expansion of the dependence of the out-of-equilibrium conductivity on the occupation number deviation.

Let us explain our claim that any perturbation will lead to enhanced conductivity. This may come about by two physically different mechanisms: first of all, we note that when the system is excited, we create vacant sites well below the Fermi surface and add electrons above it. Electrons will tunnel between these sites and, thus, even at very low temperatures current may flow through the system.

It should be mentioned that isolated relaxing modes, which are essentially those which give rise to the logarithmic relaxation of the occupation numbers, will also influence the resistance of the system. To see this, it is instructive to look at the Miller–Abrahams resistor network. The resistance of the system, in equilibrium, will be determined by a small subset of resistors that are part of the so-called backbone or percolation path. Due to the long-ranged Coulomb interactions, small changes in the mean occupation numbers far away from the backbone will change the electrostatic potential on the electronic sites belonging to the backbone and, therefore, may significantly change the values of the resistors or change the backbone geometry altogether. The formal divergence of sums of the type of Eq. (5) means that sites far away from the backbone will also be important to this process.

The percolation picture may also account for the separation of scales between the time scales involved in the transport process and those of the relaxation processes: the resistance of the system, within this picture, is determined by the bottleneck of the resistor network. Thus, the time scale associated with transport is given by the relation $R=\frac{kT}{e^2\gamma}$,[12] where R is the resistance of the bottleneck resistor, comparable to that of the system, which is $R_q e^{(T_0/T)^{1/2}}$. One may find isolated relaxing modes with much longer time scale since these are not constrained to belong to the resistor backbone, which allows for a much larger phase space. Moreover, some relaxations can occur only through multiparticle transitions, which are not considered in the current mean-field approach. These would make this time scale even longer.

The second mechanism of conductance relaxation is more subtle and is related to the Coulomb gap. Let us look at the Einstein relation,[34] $\sigma=e^2\frac{dn}{d\mu}D$. We do not expect to have any anomalies in the *thermodynamic* DOS, $\frac{dn}{d\mu}$, but the diffusion constant D should be much smaller. This is because the single-particle DOS at the chemical potential vanishes: moving an electron from a site with energy close to the chemical potential to another site will necessitate an energy of order of the width of the Coulomb gap. Therefore, the Coulomb gap may significantly lower the conductivity and, upon perturbing the system, the conductivity increases with the single-particle DOS at μ (Ref. 35) (for systems close to a local equilibrium, which exhibit the Coulomb gap). The temperature should be low enough such that the local equilibrium Coulomb gap would not be smeared.[19] Let us suppose that due to our initial perturbation of the local equilibrium configuration, we have some excess (positive or negative) δn_i in the occupation number at site i. Assuming these numbers to be random with a standard deviation δn, the energy at site j will now have an additional contribution $\Sigma_i\frac{\delta n_i}{r_{ij}}$. A finite single-particle DOS at μ will arise, which is proportional to δn. It should be noted that the sum would diverge for an infinite system due to the long-range nature of the Coulomb potential, unless correlations exist among the δn_i. Since we have not discussed the relation of δn_i to the perturbation applied, we will not address this issue here. The proportionality to δn is the crux of the matter here and suggests that within this model, the conductance relaxation should have a similar time dependence to that of the occupation number relaxation.

IV. CONCLUSIONS

We have studied a finite temperature mean-field model for the dynamics of the electron glass system. For a perturbation that drives the system not too far from the (mean-field) local equilibrium, we mapped the problem onto rate equations with a random relaxation matrix A. The matrix A belongs to a class different than the Gaussian random matrix ensembles. We found that the distribution of the eigenvalues is $\sim\frac{1}{|\lambda|}$ and naturally yields a logarithmic decay of the occupation num-

bers. This may lead to a logarithmic decay of the conductance. Such a logarithmic decay of the conductance is experimentally observed in many cases. We should stress that transitions among different metastable states are not captured within the present mean-field model and may be important for the dynamics at larger time scales. Nevertheless, we emphasize the remark made before that the $\frac{1}{\lambda}$ distribution of decay eigenvalues should be much more general than the specific model considered. It might also hold, for example, in the case of multiparticle transitions. Further research is needed to obtain additional predictions of this model, such as the time dependence of the Coulomb gap and the voltage-dependent conductance in the "two-dip" experiment.[8]

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