Problem set 11 for Quantum Field Theory course

2018.04.24.

Topics covered

- Renormalization: two-loop examples of ϕ^4

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 10.2, 10.5

Problem 11.1 Renormalization of ϕ^4 mass

We now go beyond one-loop with the renormalization of ϕ^4-theory with counter-terms:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{1}{2}m^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4 + \frac{1}{2}\delta_Z(\partial_\mu\phi_R)^2 - \frac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4, \tag{1}$$

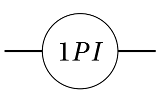
where ϕ_R is the rescaled field

$$\phi = Z^{1/2}\phi_R \tag{2}$$

with the field strength renormalization Z. λ and m are the physically measured values and δ_λ, δ_m, δ_Z are the counter terms fixed by the renormalization conditions.

(a) Draw the diagrams up to second order in λ for a propagator. Bear in mind that the δ_λ term from (1) also appears in one of these graphs, and we have contributions from δ_Z and δ_m as well.
Remark: note that this expansion is $O(\lambda^2)$, and δ_λ was computed up to this order in Problem 10.1.

(b) Let us set the renormalization conditions for δ_m and δ_Z. First, define $\Sigma(p^2)$ as being proportional to the sum of one-particle-irreducible insertions to a propagator (see Fig. 1.)



$$= -i\Sigma(p^2)$$

Figure 1: Definition of $\Sigma(p^2)$.

Then the fully dressed two point function can be written as a sum of such terms with an increasing number of consecutive 1PI insertions. Sum it up to yield the result

$$\frac{i}{p^2 - m^2 - \Sigma(p^2) + i\varepsilon} \tag{3}$$

for the interacting propagator. Set the renormalization conditions such that in any order of the perturbation theory the pole of the propagator is at $p^2 = m^2$ and also that $Z = 1$.
Hint: one can get an expression for Z by doing a Taylor expansion in $\Sigma(p^2)$ around the pole. Then the propagator is multiplied by a term (that is Z) which sets a condition for the derivative of $\Sigma(p^2)$.

(c) Start with terms of $O(\lambda)$. Compute the one-loop integral to get a result for δ_m and δ_Z up to first order:

$$\delta_Z = 0, \tag{4}$$

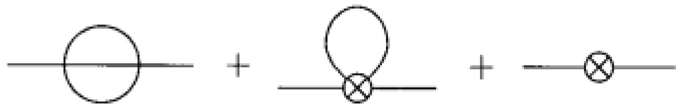


Figure 2: Second order contributions to the boson propagator. (Figure from PS.)

and

$$\delta_m = -\frac{\lambda}{2(4\pi)^{d/2}} \frac{\Lambda(1-d/2)}{(m^2)^{1-d/2}}. \tag{5}$$

(d) Write down the contribution of the "setting sun graph", i.e. one of the two second-order contributions to the propagator without a counter term (the one visible on Fig. 2.). What is the symmetry factor? What are the $O(\lambda^2)$ contributions of the rest of the graphs?

Problem 11.2 From the setting sun to field strength renormalization

In this exercise we are going to continue with the evaluation of $O(\lambda^2)$ terms in the propagator. The corresponding diagrams are shown in Fig. 2. The second two-loop diagram gives a shift to δ_m only, so let us neglect that, since we are working to obtain a formula for δ_Z.

(a) The contribution of the first term can be written as

$$\frac{(-i\lambda)^2}{6} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{q^2 - m^2 + i\varepsilon} \frac{i}{(p-k-q)^2 - m^2 + i\varepsilon}. \tag{6}$$

Combine the last two denominators and use the results of Problem 10.1 to write down the result of the q loop integral.

(b) Introduce another Feynman parameter y and show that the following expression reads

$$\frac{i\lambda^2 \Gamma(2-d/2)}{6(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\mathrm{d}y \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{[x(1-x)]^{d/2-2} (1-y)^{1-d/2} \Gamma(3-d/2)/\Gamma(2-d/2)}{\left[\left(k_E - (1-y)p_E\right)^2 + y(1-y)p_E^2 + \left(y + \frac{1-y}{x(1-x)}\right) m^2\right]^{3-d/2}}. \tag{7}$$

Hint: use the most general form of Feynman parametrization, as seen in Problem 5.1(d).

(c) Perform this integral using results from earlier calculations to get

$$\frac{i\lambda^2}{6(4\pi)^d} \int_0^1 \mathrm{d}x\mathrm{d}y \frac{\Gamma(3-d)\,[x(1-x)]^{d/2-2}\,(1-y)^{1-d/2}}{\left[y(1-y)p_E^2 + \left(y + \frac{1-y}{x(1-x)}\right) m^2\right]^{3-d}} \tag{8}$$

(d) Let us calculate the field strength renormalization δ_Z in the massless $m = 0$ limit. Show that in this case the second diagram of Fig. 2 disappears.

Hint: use results of Problem 11.1 (c) to see that it is proportional to m.

(e) Show that the third diagram contributes as

$$i\delta_Z p^2, \tag{9}$$

if $m = 0$ (cf. 11.1 (c) again). Take the limit $\epsilon = 4 - d \to 0$ to express the divergent part of δ_Z as

$$\delta_Z = -\frac{\lambda^2}{12(4\pi)^4} \frac{1}{\epsilon}. \tag{10}$$

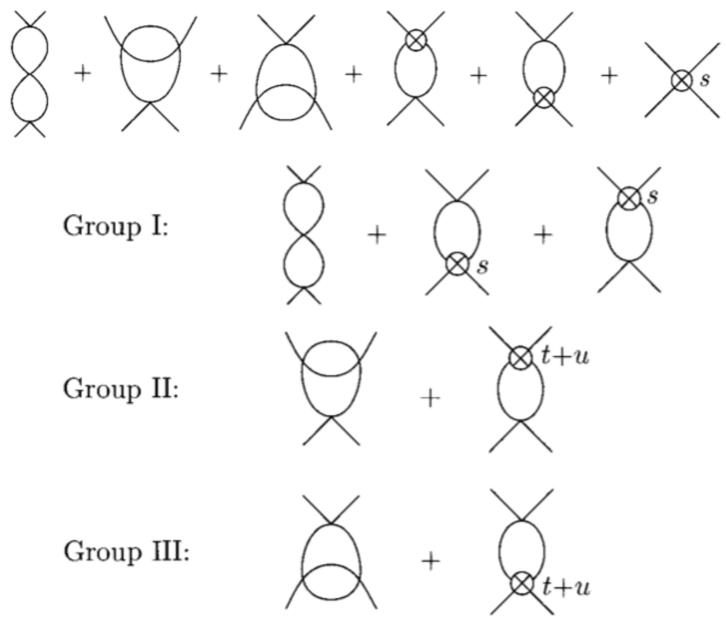


Figure 3: s-channel Feynman diagrams for the vertex function of the ϕ^4 theory at two loops.

Problem 11.3 A 2-loop calculation, Part I

The following three problems illustrate in the ϕ^4 theory that the counter terms found at 1-loop order are sufficient to renormalize the theory at all orders, even overlapping divergences. In particular, the goal is to show that all momentum dependent divergences are cancelled by the 1-loop counter terms at the 2-loop calculation of the vertex function.

There are 16 Feynman diagrams at this order. In the s-channel there are 5 independent diagrams shown in the first row of Fig. 3, there are analogous diagrams in the t and u channels. Finally, the 16$^{\text{th}}$ diagram is the 2-loop momentum independent counter term $-i\delta\lambda^{(2)}$ which gets contributions from all 3 channels, so it is convenient to split it into s, t, and u contributions. Clearly, it is sufficient to show the cancellation of divergences in the s-channel.

Recall from Problem 10.1 that up to 1-loop the 4-point function is

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2\left[iV(s) + iV(t) + iV(u)\right] - i\delta\lambda, \tag{11}$$

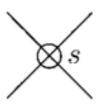
where $V(p^2)$ is the diagram with 1 loop and 4 truncated legs,

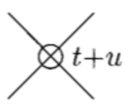
$$V(p^2) = -\frac{1}{2}\int_0^1 \mathrm{d}x \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \frac{1}{\left[m^2 - x(1-x)p^2\right]^{2-d/2}}. \tag{12}$$

From the renormalization conditions we obtained

$$-i\delta\lambda = (-i\lambda)^2\left[V(4m^2) + 2V(0)\right] + O(\lambda^3) \tag{13}$$

which we now split as



$= (-i\lambda)^2 \cdot -iV(4m^2);$


$= (-i\lambda)^2 \cdot -2iV(0).$

Now we group the 5 s-channel diagrams into 3 groups as in Fig. 3

(a) Write the sum of diagrams in Group I in terms of the $V(p^2)$ function.

(b) Using the expansion of $V(p^2)$ in Eq. (12) show that it is the sum of a finite and a *momentum independent* divergent term which can be absorbed in the new vertex counter term. What is the momentum dependence of the finite term for large momenta?

Problem 11.4 A 2-loop calculation, Part II

We continue with the calculation of the diagrams in Group II (Group III is essentially identical).

(a) Write down the expression for the first diagram in Group II exploiting that the diagram includes $V(p^2)$. (Label the incoming momenta by p_1 and p_2, the outgoing momenta by p_3 and p_4, and one of the internal momenta in the lower big loop by k.)

(b) Proceed by using the Feynman parameterization formula in Problem 5.1 (a) for the two denominators in the lower big loop to arrive at the expression

$$\text{Group II/1. diagram } = -(-i\lambda)^3 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left(k^2+2ykp+yp^2-m^2\right)^2} iV\left((k+p_3)^2\right) \tag{14}$$

(c) Substitute $V(p^2)$ from Eq. (12) and use the Feynman parameterization in Problem 5.1 (d) to join the two denominators. Show that after completing the square the result is

$$\begin{aligned}&\text{Group II/1. diagram}\\ &= -\frac{\lambda^3}{2}\frac{\Gamma(4-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}w \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{w^{1-\frac{d}{2}}(1-w)}{\left[(wx(1-x)+1-w)l^2-P^2+m^2\right]^{4-\frac{d}{2}}},\end{aligned} \tag{15}$$

where l is a shifted momentum and P^2 is a complicated expression of w, x, y, p, p_3 for which

$$P^2 \xrightarrow{w\to 0} y(1-y)p^2 + O(w)\,. \tag{16}$$

(d) Perform the Wick rotation and use the expression in Problem 5.3 (b) to obtain

$$\text{Group II/1. diagram } = -i\frac{\lambda^3}{2}\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w \frac{w^{1-\frac{d}{2}}(1-w)}{[wx(1-x)+1-w]^{\frac{d}{2}}}\frac{1}{(m^2-P^2)^{4-d}}\,. \tag{17}$$

Problem 11.5 A 2-loop calculation, Part III

Expression (17) has an obvious pole at $d=4$ coming from the Gamma function but another singularity comes from the $w=0$ boundary of the w-integral (why?). Let us separate this singularity by writing

$$\text{Group II/1. diagram } = \int_0^1 \mathrm{d}w\, w^{1-\frac{d}{2}} f(w) = \int_0^1 \mathrm{d}w\, w^{1-\frac{d}{2}}[f(w)-f(0)] + \int_0^1 \mathrm{d}w\, w^{1-\frac{d}{2}} f(0)\,. \tag{18}$$

The first term now only contains the Gamma function pole whose residue is given by the rest of the integrand at $d=4$ which is independent of momenta! This divergence is thus “eaten up” by the vertex counter term, so we turn to the second term.

(a) Perform the straightforward integrals in the second term in Eq. (18) and expand in $\varepsilon = 4-d$ to obtain

$$-i\frac{\lambda^3}{2(4\pi)^4}\frac{2}{\varepsilon}\int_0^1 \mathrm{d}y\left[\frac{1}{\varepsilon}-\gamma_{\mathrm{E}}-\log\left(\frac{m^2-y(1-y)p^2}{4\pi}\right)\right]. \tag{19}$$

Note the appearance of a double pole and a *non-local divergence* which is not polynomial in p. The double pole does not depend on the momenta so it is swallowed by the vertex counter term.

(b) The last non-trivial step is to show that the non-local divergent parts are exactly cancelled by the divergent terms of the second diagram of Group II. Write down the integral expression for this diagram (remember the definition of the $t+u$ counter term given below Eq. (13)). Expanding in ε show explicitly the cancellation of divergences.