Problem set 7 for Quantum Field Theory course

2019.04.02.

Topics covered

- Mott scattering
- Compton scattering, annihilation and pair creation
- Fermionic polarization sums
- Photonic polarization sums and Ward identity
- Massive vector boson field

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 4.8, 5.1, 5.4-5.5

Problem 7.1 Mott scattering

We return to the problem of a Dirac fermion scattering on a nucleon whose electromagnetic field described by a vector potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = e\int d^3x \overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x) \tag{1}$$

where e is the electric charge.

In Problem 6.4 it was shown that up to leading order the invariant transition matrix element is given by

$$i\mathcal{M} = -ie\bar{u}^{s'}(\mathbf{p}')\gamma^\mu u^s(\mathbf{p})\tilde{A}_\mu(q)\,, \tag{2}$$

with $q = p' - p$. We are going to work out the solution for the scattering cross-section in the ultrarelativistic limit.

(a) Taking the z-axis along the direction of $\mathbf{p}$ ie. $\mathbf{p} = (0,0,p_3)$ with $p_3 > 0$, the incoming helicity eigenstates are given by

$$u^s(\mathbf{p}) = \mathcal{N}\begin{pmatrix}\chi^s \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_\mathbf{p}+m}\chi^s\end{pmatrix} = \mathcal{N}\begin{pmatrix}\chi^s \\ \frac{p_3\sigma_3}{E_\mathbf{p}+m}\chi^s\end{pmatrix}, \tag{3}$$

where $\mathcal{N} = \sqrt{E_\mathbf{p}+m}$ and $\chi^\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi^\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$ has spin-z equal to $\pm 1/2$, respectively. Rewrite the Dirac spinors in terms of

$$\eta = \frac{|\mathbf{p}|}{E_\mathbf{p}+m} = \frac{p_3}{E_\mathbf{p}+m}\,, \tag{4}$$

so we have $\eta \to 0$ for the non-relativistic case and $\eta \to 1$ corresponds to an ultrarelativistic particle.

For the outgoing particle with $\mathbf{p}' = |\mathbf{p}'|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ pointing in the direction given by the solid angle (θ,ϕ), the spin projection basis along $\mathbf{p}'$ is given by

$$\xi^\uparrow = \begin{pmatrix}\cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2}\end{pmatrix},\ \xi^\downarrow = \begin{pmatrix}-\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2}\end{pmatrix} \tag{5}$$

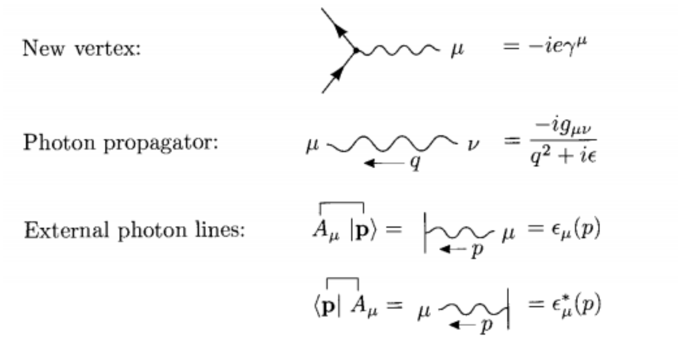

Figure 1: Feynman rules for interactions involving the photon. The metric tensor is denoted by $g_{\mu\nu}$.

Check that they are really helicity projections, i.e. they satisfy

$$\vec{p}\,' \cdot \vec{\sigma}\, \xi^{\uparrow,\downarrow} = \pm|\mathbf{p}'|\xi^{\uparrow,\downarrow} \tag{6}$$

and write the corresponding helicity basis for the outgoing spinor amplitudes as

$$u^s(\mathbf{p}') = \mathcal{N} \begin{pmatrix} \xi^s \\ \frac{\pm|\mathbf{p}'|}{E_{\mathbf{p}'}+m}\xi^s \end{pmatrix}, \tag{7}$$

(b) We saw in Problem 6.4 that we can describe the scattering in terms of θ scattering angle. Use a frame in which incoming and outgoing particles are in the helicity basis written above.
Now calculate all possible spin polarized currents

$$\bar{u}^{\uparrow/\downarrow}(\mathbf{p}')\gamma^\mu u^{\uparrow/\downarrow}(\mathbf{p}). \tag{8}$$

Show that in the ultrarelativistic limit $\eta \to 1$ the helicity is conserved!

(c) Turning to the cross-section, we found earlier that for this process it can be expressed as

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2\Big|_{|\mathbf{p}|=|\mathbf{p}'|}. \tag{9}$$

Assuming an unpolarized incoming electron beam we must take a spin average, while if the outgoing polarisation is undetected it must be summed over, so the total transition probability can be obtained as

$$|\mathcal{M}|^2 = \frac{1}{2}\sum_{s_i,s_f}|\mathcal{M}(e^-_{s_i} \to e^-_{s_f})|^2, \tag{10}$$

where s_i, s_f are the polarisations for the incoming and outgoing particles, respectively. Utilize your previous results at $\eta = 1$ to get the ultrarelativistic scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E^2\sin^4(\theta/2)}\cos^2(\theta/2). \tag{11}$$

Hint: Recall that the Coulomb potential is expressed as $A_0(\mathbf{q}) = Ze/q^2$ in momentum space and there is no magnetic field.

Problem 7.2 Elementary QED processes and crossing symmetry

The Feynman rules for photonic terms of QED in momentum space are shown in Fig. 1 (figure from Peskin&Schroeder). For fermionic rules see Fig. 1. of Problem set 6.

(a) Draw the leading-order diagrams for Compton scattering, i.e. for a $e^-\gamma \to e^-\gamma$ process. What is the corresponding amplitude?
Hint: Label momenta as follows: p and p' for electrons and k, k' for photons. Note that there are two distinct ways to contract external photon legs with the vertices.

(b) Draw the leading-order diagrams for the annihilation of an electron and a positron: $e^- e^+ \to 2\gamma$. Write the amplitude for this process as well.

(c) The above results show a nice illustration of the so-called crossing symmetry. In terms of diagrams, it manifests as a $-\pi/2$ rotation. What is the corresponding transformation rule for the set of momenta?

(d) A $+\pi/2$ rotation of Compton scattering diagrams results in leading order diagrams for pair creation. Write down the amplitude for this process directly using crossing symmetry .

(e) Application of crossing symmetry is straightforward at the level of amplitudes. Cross-section computations, however, involve spin sums:

$$\sum_s u^s(p)\bar{u}^s(p)\,. \tag{12}$$

What is the result if we naively apply the transformation rule of momenta corresponding to crossing symmetry?

Problem 7.3 Fermionic polarization sums

Let us consider the annihilation of an electron-positron pair to a muon-antimuon pair $e^- e^+ \to \mu^- \mu^+$.

(a) Draw the leading order diagram for this process, and write down the corresponding amplitude.
Hint: we can apply the same set of fermionic Feynman rules for the muon except that its mass is m_μ instead of m_e.

(b) When computing scattering cross-sections we have to take absolute value squared of this amplitude. If the incoming beam is unpolarized we must take the average:

$$\frac{1}{2}\sum_s\,,\ s\in \text{in}\,. \tag{13}$$

while if using a detector that is insensitive to polarisation, outgoing spins must be summed over:

$$\sum_r\,,\ r\in \text{out}\,. \tag{14}$$

Utilizing these prescriptions, write down the formula for the spin average of the squared amplitude.
Hint: you can treat the electronic and muonic part separately, since they are scalars in spinor space. Then write the products with explicit spinor indices and use the known spinor sum rules.

(c) Exploit the trace identities (Problem 3.2(b)) to bring the expression to the form:

$$|\mathcal{M}|^2_{\text{avg}} = \frac{8e^4}{q^4}(p'^{\mu}p^{\nu} - (pp')\eta^{\mu\nu} + p^{\mu}p'^{\nu} - m_e^2\eta^{\mu\nu})(k'_\mu k_\nu - (kk')\eta_{\mu\nu} + k_\mu k'_\nu - m_\mu^2\eta_{\mu\nu})\,, \tag{15}$$

using the notation p, p' for electronic and k, k' for muonic momenta and $q = p + p'$.

(d) Considering that $m_\mu \gg m_e$ it is a good approximation to set $m_e = 0$. Working in this limit, derive the nice expression

$$|\mathcal{M}|^2_{\text{avg}} = \frac{8e^4}{q^4}\left[2(pk')(p'k) + 2(pk)(p'k') + 2m_\mu^2(pp')\right]. \tag{16}$$

Problem 7.4 Photon polarisation sums

Many elementary QED processes have external photon legs. Consequently, polarisation four-vectors $\varepsilon_\mu(k)$ appear in amplitudes of such processes. When calculating cross-section, we often must perform polarisation sums of photon legs as well (for the fermionic counterpart, see Problem 7.3).

(a) Consider a process with an outgoing external photon leg with momentum k. Its invariant amplitude can be written as

$$i\mathcal{M} = i\varepsilon^*_\mu(k)\mathcal{M}^\mu(k)\,. \tag{17}$$

The polarisation sum is then expressed as

$$|\mathcal{M}|^2 = \sum_{\lambda=1}^{2} |\varepsilon^{(\lambda)*}_\mu(k)\mathcal{M}^\mu(k)|^2 = \sum_{\lambda=1}^{2} \varepsilon^{(\lambda)*}_\mu(k)\varepsilon^{(\lambda)}_\nu(k)\mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k)\,, \tag{18}$$

since only physical polarisations can occur in outgoing states.
Adapting our usual frame $k^\mu = (k,0,0,k)$ and $\varepsilon^{(1)\mu} = (0,1,0,0)$, $\varepsilon^{(2)\mu} = (0,0,1,0)$ write the polarisation sum explicitly.

(b) Recall that the photon is created by the $j_\mu A^\mu$ term, where j^μ is the Dirac current. Then we can write the amplitude in question as

$$\mathcal{M}^\mu(k) = \int d^4x e^{ikx} \langle f|\, j^\mu(x)\,|i\rangle\,. \tag{19}$$

Show that assuming this form the Ward identity is true, i.e.

$$k_\mu \mathcal{M}^\mu = 0\,. \tag{20}$$

(c) Use the Ward identity (taking advantage of the special frame) to show that we can replace

$$\sum_{\lambda=1}^{2} \varepsilon^{(\lambda)*}_\mu(k)\varepsilon^{(\lambda)}_\nu(k) \longrightarrow -\eta_{\mu\nu}\,. \tag{21}$$

(d) Consider the amplitude of Compton scattering:

$$i\mathcal{M} = -ie^2\varepsilon^*_\mu(k')\varepsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p)\,. \tag{22}$$

Show by explicit calculation that it satisfies the Ward identity, i.e. we get zero if we replace $\varepsilon^\mu(k')$ with k'^μ and similarly for $\varepsilon^\nu(k)$ replaced with k^ν.
Hint: a clever insertion of the momentum space Dirac equation does the trick.

Problem 7.5 Physics of a massive vector boson field

The Lagrangian density of a charged massive vector field can be written as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu W^\dagger_\nu - \partial_\nu W^\dagger_\mu)(\partial^\mu W^\nu - \partial^\nu W^\mu) + M_W^2 W^\dagger_\mu W^\mu\,. \tag{23}$$

(a) Write down the Euler-Lagrange equations of motion. Take the divergence to show that $\partial_\mu W^\mu = 0$, and that the equations can be reduced to a simpler form.

(b) After quantisation we can write the following plane-wave expansion:

$$W_\mu = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda=1}^{3}\left[a^\lambda(\mathbf{k})\varepsilon^\lambda_\mu(\mathbf{k})e^{-ikx} + b^{\lambda\dagger}(\mathbf{k})\varepsilon^{\lambda*}_\mu(\mathbf{k})e^{ikx}\right]. \tag{24}$$

Remark: from $\partial_\mu W^\mu = 0$ it is easy to see that there are only 3 independent polarisation vectors.
Show that the following quantisation conditions:

$$\left[a^\lambda(\mathbf{k}), a^{\lambda'\dagger}(\mathbf{k}')\right] = (2\pi)^3\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{k}'),\quad \left[b^\lambda(\mathbf{k}), b^{\lambda'\dagger}(\mathbf{k}')\right] = (2\pi)^3\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{25}$$

(with all other commutators vanishing), combined with the completeness relations of the polarisations ε:

$$\sum_{\lambda=1}^{3}\varepsilon^\lambda_\mu(\mathbf{k})\varepsilon^{\lambda*}_\nu(\mathbf{k}) = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2} \tag{26}$$

lead to the following form for the equal-time commutation relations:

$$\left[W_\mu^\dagger(t,\mathbf{x}), \dot{W}_\nu(t,\mathbf{y})\right] = -i\left(\eta_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{M_W^2}\right)\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{27}$$

where the differential operator projects the canonical commutation relations to the subspace which is Lorentz orthogonal to the momentum. This is a consequence of the divergence constraint on the field: note that the 4×4 matrix

$$P_\mu^{\ \nu}(k) = -\delta_\mu^{\ \nu} + \frac{k_\mu k^\nu}{M_W^2} \tag{28}$$

is a projector on the subspace Lorentz orthogonal to k, i.e. it satisfies

$$P(k)^2 = P(k) \quad \text{and} \quad P_\mu^{\ \nu}(k)k_\nu = 0\,. \tag{29}$$

(c) Apart from the two transversal polarisations the presence of nonzero mass also allows for a physical longitudinal polarization. Write the corresponding polarisation vector as

$$\varepsilon_\mu^3(\mathbf{k}) = \mathcal{N}_0(\varepsilon_0, \mathbf{k})\,. \tag{30}$$

Imposing the orthonormality of polarisation vectors

$$\varepsilon_\mu^\lambda(\mathbf{k})\varepsilon^{\mu\lambda'}(\mathbf{k}) = -\delta^{\lambda\lambda'}, \tag{31}$$

combined with divergence condition, calculate $\mathcal{N}_0$ and ε_0.

(d) * Compute the commutator function

$$\left[W_\mu(x), W_\nu^\dagger(y)\right] = ? \tag{32}$$

and the Feynman propagator

$$\langle 0|T W_\mu(x) W_\nu^\dagger(y)|0\rangle = i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-\eta_{\mu\nu} + k_\mu k_\nu/M_W^2}{k^2 - M_W^2 + i\varepsilon} - \frac{2i}{M_W^2}\delta_{\mu 0}\delta_{\nu 0}\delta^{(4)}(x-y) \tag{33}$$

Remark: the presence of the non-covariant last term may seem surprising at first, but its effects can be shown to cancel, leaving only Lorentz-invariant physical amplitudes.