Problem set 5 for Quantum Field Theory course

2019.03.12.

Topics covered

- Feynman integral techniques: Wick rotation and Feynman parametrisation
- Dimensional regularisation
- Wick's theorem

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 4.3, 4.5

Problem 5.1 Feynman parametrisation

Recall the form of Feynman propagator for a Dirac field

$$S_{\mathrm{F}}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip(x-y)} \,. \tag{1}$$

The Feynman perturbative expansion in interacting field theories often involves integrals containing the product of two (or more) propagators, e.g.

$$\int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\epsilon} \,. \tag{2}$$

The following exercises develop some useful techniques to handle this kind of integrals.

(a) The first step is to combine the two denominators. The so-called Feynman parametrisation utilizes the following identity:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2} = \int_0^1 dx \int_0^1 dy\, \delta(x+y-1) \frac{1}{(xA+yB)^2} \,. \tag{3}$$

Prove this equation by explicit calculation.

(b) Show that taking derivative of both sides with respect to B $n-1$ times yields

$$\frac{1}{AB^n} = \int_0^1 dx \int_0^1 dy\, \delta(x+y-1) \frac{ny^{n-1}}{(xA+yB)^{n+1}} \,. \tag{4}$$

(c) One can also generalise (3) in a different way:

$$\prod_{i=1}^{n} \frac{1}{A_i} = \prod_{i=1}^{n} \left(\int_0^1 dx_i \right) \delta\left(\sum_{i=1}^{n} x_i - 1 \right) \frac{(n-1)!}{\left(\sum_{i=1}^{n} x_i A_i\right)^n} \,. \tag{5}$$

Prove this formula by induction starting from (3).

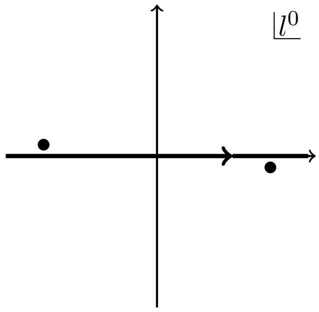

Figure 1: Integration line and placement of poles on complex l^0 plane.

(d) Using the derivative trick introduced above, derive the most general case of Feynman parametrisation:

$$\prod_{i=1}^{n} \frac{1}{A_i^{m_i}} = \prod_{i=1}^{n} \left(\int_0^1 dx_i \right) \delta \left(\sum_{i=1}^{n} x_i - 1 \right) \frac{\Gamma\left(\sum_{i=1}^{n} m_i\right)}{\prod_{i=1}^{n} \Gamma(m_i)} \frac{\prod_{i=1}^{n} x_i^{m_i - 1}}{\left(\sum_{i=1}^{n} x_i A_i\right)^{\sum_{i=1}^{n} m_i}} . \tag{6}$$

Problem 5.2 Wick rotation

Consider the following integral:

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p+k)^2 - m^2 + i\epsilon} . \tag{7}$$

(a) Rewrite the expression using identity (3) (use the single-variable version). Introducing a new variable $l^\mu = k^\mu + x p^\mu$, complete the square to obtain:

$$\int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i\epsilon)^2} , \tag{8}$$

with $\Delta = m^2 - x(1-x)p^2$.

(b) Recall that ϵ corresponds to a slight displacement of poles from the integration line that coincides with the real axis (see Figure 1). Use this to argue that integration along the real axis and integration along the imaginary axis yield the same result. This operation is called Wick rotation.

(c) For the integral along the imaginary l^0 axis one can introduce a new variable l^4. The Minkowski length becomes Euclidean $l^2 = -l_E^2$ with $l_E^2 = (l^1)^2 + (l^2)^2 + (l^3)^2 + (l^4)^2$ so the integral is now rotationally invariant and one can easily take the $\epsilon \to 0$ limit:

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i\epsilon)^2} = \frac{i}{(2\pi)^4} \int d^4 l_E \frac{1}{\left(l_E^2 + \Delta\right)^2} = \frac{i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dl_E \frac{l_E^3}{\left(l_E^2 + \Delta\right)^2} , \tag{9}$$

where $d\Omega_4$ is the surface area element for the 4-dimensional unit sphere.

Unfortunately, this integral is divergent. Replace the second power in the denominator with a general power m and calculate this integral. For what m does it converge?

(Hint: Use a variable transform $s = \frac{\Delta}{l_E^2 + \Delta}$ *and the indentity*

$$\int_0^1 s^{x-1} (1-s)^{y-1} ds = B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} , \tag{10}$$

where $B(x,y)$ *is Euler's beta function.)*

(d) Return to the integral

$$\int_0^\infty dl_E \frac{l_E^3}{(l_E^2+\Delta)^m}\,. \tag{11}$$

A common method for regularization is replacing the upper limit ∞ with a cutoff Λ. With the same variable transform as above, perform this integral explicitly in two cases $m = 2, 3$. What is the cutoff-dependence in the limit $\Lambda \to \infty$?

Problem 5.3 Dimensional regularisation

Problem 5.2(c) concluded with the divergent integral

$$I = \frac{i}{(2\pi)^4}\int d\Omega_4 \int_0^\infty dl_E \frac{l_E^{(4-1)}}{(l_E^2+\Delta)^2}\,. \tag{12}$$

An effective way to deal with such divergences is provided by dimensional regularisation. If this integral were defined in dimension $d < 4$, it would converge.

(a) Write the above integral in $d \in \mathbb{R}$ dimensions:

$$I = \frac{i}{(2\pi)^d}\int d\Omega_d \int_0^\infty dl_E \frac{l_E^{(d-1)}}{(l_E^2+\Delta)^m}\,. \tag{13}$$

Derive the following formula for $\Omega_d = \int d\Omega_d$:

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{14}$$

(Hint: Use the identity $(\sqrt{\pi})^d = \left(\int_0^\infty dx e^{-x^2}\right)^d$ and write the latter as a rotationally invariant d-dimensional integral. Use the definition of $\Gamma(z)$.)

(b) Continue the evaluation of integral (13) utilizing the hint in Problem 5.2(c). Derive the result

$$I = \frac{i}{(4\pi)^{d/2}}\Delta^{d/2-m}\frac{\Gamma(m-d/2)}{\Gamma(m)}. \tag{15}$$

Notice that the result is proportional to $\Gamma(0)$ for $d = 4$ and $m = 2$. Using $\Gamma(z+1) = z\Gamma(z)$ show that the Gamma function has simple poles if z is a non-positive integer.

(c) Perform similar steps to evaluate I_2

$$I_2 = i\int \frac{d^d l_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^m} \tag{16}$$

with the result

$$I_2 = i\frac{d/2}{(4\pi)^{d/2}}\Delta^{d/2-m+1}\frac{\Gamma(m-d/2-1)}{\Gamma(m)}\,. \tag{17}$$

(Hint: use recursive relations of $\Gamma(z)$.)

(d) Turning back to the divergence of I, consider the case where $m = 2$ and $d = 4-\epsilon$. An expansion of $\Gamma(\epsilon)$ for small ϵ yields:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon)\,, \tag{18}$$

where $\gamma_E = \lim_{n\to\infty}\left(\sum_{i=1}^n \frac{1}{k} - \ln(n)\right) = 0.5772...$ is the Euler–Mascheroni constant.
Using this result derive the following formula for I $(m = 2)$:

$$I = \frac{i}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma_E - \ln\frac{\Delta}{4\pi}\right) + O(\epsilon)\,. \tag{19}$$

(Hint: do not forget to expand the power of $\frac{\Delta}{4\pi}$ in terms of ϵ.)

(e) Our result (19) involves the logarithm of a dimensionful quantity Δ which reveals a loophole of the above calculation. This is due to the fact that dimensional regularisation changes the units of the integration measure:

$$\left[d^4 l\right] \neq \left[d^d l\right] . \tag{20}$$

This can be cured by introducing μ with $[\mu] = 1$ and replacing

$$d^4 l \longrightarrow \mu^{4-d} d^d l . \tag{21}$$

Show that this choice leads to a dimensionless argument of the logarithmic function

$$\ln \frac{\Delta}{4\pi\mu^2} . \tag{22}$$

Remark: dimensional regularisation is a powerful tool in identifying and handling divergences. Apparently, it works fine for evaluating integrals. Where can it cause problems that we replace $d = 4$ with $d \in \mathbb{R}$?

Problem 5.4 Wick's theorem and scalar fields

Consider a theory with the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\nu \Phi\right)^2 - \frac{1}{2} M^2 \Phi^2 + \frac{1}{2}\left(\partial_\nu \varphi\right)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\mu}{2} \Phi \varphi \varphi \tag{23}$$

of two real scalar fields and their interaction with strength μ. In the interaction picture they can be expanded as usual

$$\begin{pmatrix} \hat{\Phi} \\ \hat{\varphi} \end{pmatrix}(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[\begin{pmatrix} \hat{a}_{\mathbf{k}} \\ \hat{b}_{\mathbf{k}} \end{pmatrix} e^{-ikx} + \begin{pmatrix} \hat{a}^\dagger_{\mathbf{k}} \\ \hat{b}^\dagger_{\mathbf{k}} \end{pmatrix} e^{ikx} \right] , \tag{24}$$

where the creation and annihiliation operators satisfy canonical commutation relations.

The normalization condition for the interacting vacuum can be written as

$$1 = \langle \Omega | \Omega \rangle = \mathcal{N} \langle 0 | T \exp\left(-i \int d^4 x \mathcal{H}_I \right) | 0 \rangle , \tag{25}$$

where T is time-ordering operator and

$$\mathcal{H}_I = \frac{\mu}{2} \Phi \varphi \varphi . \tag{26}$$

(a) Perform a perturbative expansion and calculate $\mathcal{N}$ up to second order in μ.

(b) Use Wick's theorem to show that only even order terms contribute to $\mathcal{N}$.

(c) Now consider a state that consisting of a single Φ particle with momentum q,

$$|\mathbf{q}\rangle_\Phi = \sqrt{2E_{\mathbf{q}}} \hat{a}^\dagger_{\mathbf{q}} |0\rangle . \tag{27}$$

Calculate pairing of $\Phi(x)$ with this state:

$$\overline{\Phi(x) \, |\mathbf{q}}\rangle_\Phi = \Phi^{(+)}(x) \, |\mathbf{q}\rangle_\Phi = e^{-iqx} |0\rangle , \tag{28}$$

where

$$\Phi^{(+)}(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} a_{\mathbf{k}} e^{-ikx} \tag{29}$$

is the positive energy part of $\Phi(x)$ field. Normal ordering ensures that positive energy parts are always placed on the right.

Similarly, outgoing particle states has to be contracted with negative energy part of the field, which are always placed to the left under normal ordering. What is the value of the pairing

$$\contraction{{}_{\Phi}\langle}{\mathbf{q}}{|\,}{\Phi}{}_{\Phi}\langle\mathbf{q}|\,\Phi(x) = {}_{\Phi}\langle\mathbf{q}|\,\Phi^{(-)}(x)\,, \tag{30}$$

where

$$\Phi^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}a^{\dagger}_{\mathbf{k}}e^{+ikx} \tag{31}$$

is the negative energy part of the $\Phi(x)$ field?

(Remark: similar relations are valid for $\varphi(x)$.)

(d) Let us assume $M > 2m$, i.e. the decay of Φ to 2φ is kinematically allowed. Calculate the amplitude of such a process up to first order in μ, i.e. the following matrix element:

$${}_{\varphi}\langle\mathbf{p}_1\mathbf{p}_2| - i\frac{\mu}{2}T\int d^4x\Phi(x)\varphi(x)\varphi(x)\,|\mathbf{q}\rangle_{\Phi}\,. \tag{32}$$

Hint: be careful with possible number of pairings! What is the symmetry factor?

Argue that this result is true up to second order. Given this set of initial and final states is it possible for any even order of μ to contribute?

(e) Now consider scattering of two "light" particles on each other:

$${}_{\varphi}\langle\mathbf{p}',\mathbf{k}'|\,T\exp\left(-i\frac{\mu}{2}\int d^4x\Phi\varphi\varphi\right)|\mathbf{p},\mathbf{k}\rangle_{\varphi}\,. \tag{33}$$

Compute this matrix element up to second order in μ. What is the amplitude of such a process?

Note: only take into account fully connected terms, i.e. where all external particles are connected to a vertex inside.

Problem 5.5 Dimensional regularisation - tensors and Dirac matrices

Dimensional regularisation is the redefinition of the integral measure

$$\int\frac{d^4p}{(2\pi)^4} \longrightarrow \int\frac{d^dp}{(2\pi)^d}\mu^{4-d}\,, \tag{34}$$

with $d \in \mathbb{R}$.

(a) Consider an integrand which is a two-index tensor (that would be a Lorentz tensor prior to Wick rotation) multiplied with a scalar:

$$I^{ij} = \int\frac{d^d\mathbf{p}}{(2\pi)^d}p^ip^jf(\mathbf{p}^2)\,. \tag{35}$$

By separating rotationally noninvariant parts:

$$I^{12} = \int_{-\infty}^{\infty}dp_1dp_2p_1p_2\int\frac{d^{d-2}\mathbf{p}_{\perp}}{(2\pi)^d}f(p_1^2+p_2^2+\mathbf{p}_{\perp}^2) = 0 \tag{36}$$

(where we denoted vector components with lower indices in order to ease notations) we see that due to parity properties the integral is proportional to the Kronecker delta. Prove that the result is

$$I^{ij} = \frac{\delta^{ij}}{d}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\mathbf{p}^2f(\mathbf{p}^2)\,. \tag{37}$$

Hint: utilize (14) in $\mathbf{p}_{\perp}$ integral and perform the following variable transform: $p_{\perp}^2 = xp^2$ and $p_1^2 = (1-x)p^2$ with x running from 0 to 1 and p from 0 to ∞.

(b) *One can extend the above reasoning for tensors with multiple indices. What is the result? What changes if we do not perform Wick rotation and work with Lorentz indices μ and ν in place of i and j?

(c) Replacing $d_0 = 4$ with an arbitrary $d \in \mathbb{R}$ also affects Dirac matrices as they are representations of the Clifford algebra, which so far we only worked out in four space-time dimensions. In general, representations of the Clifford algebra satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1 \tag{38}$$

and hermiticity

$$\gamma^{\mu\dagger} = \gamma_\mu = \begin{cases} \gamma^\mu, & \text{if } \mu = 0 \\ -\gamma^\mu, & \text{if } \mu \geq 1\,, \end{cases} \tag{39}$$

The generalisation of Clifford's theorem to an even integer dimension $d = 2n$ states that the irreducible representation of Clifford algebra has 2^n dimensions and is unique up to similarity transformation (i.e. change of the basis in the spinor space).

These constraints can be satisfied using the following construction. Assume that we have a 2^n dimensional representation $\gamma^\mu_{(n)}$ with $0 \leq \mu \leq 2n-1$. In order to get a 2^{n+1}-dimensional representation (for $2n+2$ space-time dimensions) we first construct $\gamma^\mu_{(n+1)}$ for $\mu \in [0, 2n-1]$ as

$$\gamma^\mu_{(n+1)} = \begin{pmatrix} \gamma^\mu_{(n)} & 0 \\ 0 & \gamma^\mu_{(n)} \end{pmatrix}\,, \tag{40}$$

Then we define

$$\hat{\gamma}_{(n)} = i^{n-1}\gamma^0_{(n)} \cdots \gamma^{2n-1}_{(n)}\,. \tag{41}$$

Show that

$$\hat{\gamma}^\dagger_{(n)} = \hat{\gamma}_{(n)}\,, \quad \hat{\gamma}^2_{(n)} = 1\,, \quad \{\hat{\gamma}_{(n)}, \gamma^\mu_{(n)}\} = 0\,. \tag{42}$$

The γ-matrices for the remaining two μ indices can be introduced as

$$\gamma^{2n}_{(n+1)} = \begin{pmatrix} 0 & \hat{\gamma}_{(n)} \\ -\hat{\gamma}_{(n)} & 0 \end{pmatrix}\,, \quad \gamma^{2n+1}_{(n+1)} = \begin{pmatrix} 0 & i\hat{\gamma}_{(n)} \\ i\hat{\gamma}_{(n)} & 0 \end{pmatrix}\,. \tag{43}$$

Show that this also satisfies (38) and (39), hence we have a representation for any even space-time dimension $d = 2n$.

(d) Defining $\eta^\mu_\mu = d$ prove the following identities:

$$\gamma^\mu\gamma_\mu = d1\,, \quad \gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu \tag{44}$$

(e) For Feynman graph calculations trace identities involving γ matrices are also needed. The analytic continuation of the trace operation for any $d \in \mathbb{R}$ is defined to be linear and invariant under cyclic permutation. d-dependence is only in trace of identity matrix

$$\operatorname{tr} 1 = f(d)\,. \tag{45}$$

Calculate the trace of an even number of γs and for an odd number of them generalizing results of Problem 3.2.