Problem set 1 for Quantum Field Theory course

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Topics covered

- Dirac equation: spinors and Clifford algebra
- Classical field theory: Euler–Lagrange equation and Noether currents

Recommended reading G. Takács: Lecture notes for Particle Physics MSc course, the following chapters:

- Chapter 5.1-5.3, 5.5
- Chapter 8
- Chapter 9

Problem 1.1 Spinor transformations

The Dirac equation has the following form:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0, \tag{1}$$

where the γ^μ are 4×4 matrices satisfying the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu}\mathbf{1}_{4\times 4}, \tag{2}$$

and Ψ is a four-component Dirac spinor. Under a Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{3}$$

Dirac spinors transform as

$$\Psi'(x') = S(\Lambda)\Psi(x). \tag{4}$$

(a) Show that

$$S(\Lambda)\gamma^\mu S(\Lambda)^{-1} = \left(\Lambda^{-1}\right)^\mu{}_\nu \gamma^\nu. \tag{5}$$

(Hint: exploit the fact that Eq. (1) *must be invariant under Lorentz transformations.)*

(b) Show that Lorentz transformed γ-matrices $\gamma'^\nu = \left(\Lambda^{-1}\right)^\nu{}_\mu \gamma^\mu$ satisfy (2).

Problem 1.2 Group structure and generators

(a) Prove that $S(\Lambda_1)S(\Lambda_2)$ solves the fundamental relation (5) for $\Lambda_1\Lambda_2$.

An infinitesimal Lorentz transformation is of the form

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \varepsilon\omega^\mu_\nu + O(\varepsilon^2), \tag{6}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. In the spinor representation it is represented as

$$S(\Lambda) = \mathbf{1} - \frac{i}{4}\varepsilon\, \omega_{\mu\nu}\sigma^{\mu\nu} + O(\varepsilon^2), \tag{7}$$

so spinor transformations are generated by $\sigma^{\mu\nu}$.

(b) Prove that

$$[\gamma^\lambda, \sigma^{\mu\nu}] = 2i\left(\eta^{\lambda\mu}\gamma^\nu - \eta^{\lambda\nu}\gamma^\mu\right). \tag{8}$$

(c) Prove that $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ satisfies Eq. (8).
Hint: Make use of the Jacobi identity $[A,[B,C]] = \{\{A,B\},C\} - \{B,\{A,C\}\}$.

Problem 1.3 Lagrangian density

Equation (1) is the Euler–Lagrange equation for the following Lagrangian density

$$\mathcal{L} = \overline{\Psi}(i\not{\partial} - m)\Psi, \tag{9}$$

where $\not{\partial} = \gamma^\mu \partial_\mu$ and $\overline{\Psi} = \Psi^\dagger \gamma^0$. Show that the Lagrangian density is not real:

$$\mathcal{L}^* = \overline{\Psi}(-i\overleftarrow{\not{\partial}} - m)\Psi. \tag{10}$$

Show that the difference between $\mathcal{L}$ and $\mathcal{L}^*$ is a total four-divergence, and so one can introduce a real Lagrangian density

$$\mathcal{L}' = \frac{1}{2}\left(\mathcal{L} + \mathcal{L}^*\right) \tag{11}$$

which differs from (9) by a total four-divergence.

Problem 1.4 Euler–Lagrange equations

The equations of motion for a field can be obtained from its Lagrangian density using the Euler–Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \Phi_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} = 0, \tag{12}$$

where α labels components of the fields.

(a) Obtain the Klein–Gordon equation from the Lagrangian density of a complex scalar field

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi. \tag{13}$$

Note: this can be done in two ways: first, in term of fields φ_1 and φ_2 where $\Phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$. Second, choosing our two fields as Φ and Φ^*. Show that both approaches lead to the same equation!

(b) Derive the Dirac equation (1) and its conjugate from the Lagrangian (9). *(Hint: exploit variational independence of Ψ and $\overline{\Psi}$.)* Check that you obtain the same equations from (11).

(c) Obtain Maxwell's equations from the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{14}$$

expressed in terms of the four-vector potential A_μ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(d) Derive the equations of motion for a massive vector field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \tag{15}$$

Prove that the four-divergence of A^μ vanishes if $m^2 \neq 0$! *(Hint: calculate four-divergence of the Euler–Lagrange equation!)*

Making use of the above fact prove that each component A^ν satisfies a Klein–Gordon equation:

$$(\partial_\mu \partial^\mu - m^2)A^\nu = 0. \tag{16}$$

Problem 1.5 Noether currents and energy-momentum tensor

An infinitesimal transformation of the fields

$$\Phi'_\alpha(x) = \Phi_\alpha(x) + \varepsilon\delta\Phi_\alpha(x) + O(\varepsilon^2) \tag{17}$$

leaves the equation of motion invariant (i.e. it's a symmetry) if it changes the Lagrangian density by a total divergence

$$\mathcal{L}' = \mathcal{L} + \varepsilon\partial_\mu K^\mu + O(\epsilon^2). \tag{18}$$

Noether's theorem states that the following current is conserved:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\delta\Phi_\alpha - K^\mu. \tag{19}$$

If we consider a system invariant under global translations:

$$x'^\mu = x^\mu + \varepsilon^\mu, \tag{20}$$

then the conserved current can be written in the form

$$j^\mu = -\varepsilon_\nu T^{\nu\mu}, \tag{21}$$

where $T^{\mu\nu}$ is called the canonical energy-momentum tensor. It can be expressed by transformation of the fields:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)}\partial^\nu\Phi_\alpha - \eta^{\mu\nu}\mathcal{L}. \tag{22}$$

(a) Show that energy-momentum tensor can be redefined using

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu}, \tag{23}$$

where $B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}$. Use the conservation of Noether current in (21).

Note: the above transformation is called a Belinfante–Rosenfeld redefinition. The freedom in definition results from the fact that only energy and momentum can be measured direclty, but not the corresponding currents. Therefore to show the validity of the redefinition it is necessary to show that (i) it leaves the conserved quantities invariant and (ii) it also leaves the continuity equation unchanged.

(b) Compute the energy-momentum tensor for a scalar field with Lagrangian density (13).

(c) Calculate $T^{\mu\nu}$ from the Lagrangian density $\mathcal{L}'$ of the Dirac field and show that it is real. What if we computed $T^{\mu\nu}$ from $\mathcal{L}$ of (9) instead?

(d) Derive the energy-momentum tensor of electrodynamics (Eq. (14)) with $j^\mu = 0$. Unfortunately, the result is not gauge invariant. Correct for this by applying a BR transformation with

$$B^{\lambda\mu\nu} = F^{\lambda\mu}A^\nu, \tag{24}$$

and show that $\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu}$ is the energy-momentum tensor obtained in classical electrodynamics with the components

$$\begin{aligned}
\Theta^{00} &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), &(25)\\
\Theta^{0j} &= (\mathbf{E}\times\mathbf{B})^j = S^j, &(26)\\
\Theta^{ij} &= -E^iE^j + \frac{1}{2}\delta^{ij}\mathbf{E}^2 - B^iB^j + \frac{1}{2}\delta^{ij}\mathbf{B}^2. &(27)
\end{aligned}$$

(e) What is $T^{\mu\nu}$ for a massive vector field (15)? What $\Theta^{\mu\nu}$ do we obtain applying the same transformation as for the electromagnetic field?

Problem 1.6 $U(1)$ Noether current for Dirac field

The (real) Lagrangian density of the Dirac fermion can be written as

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m\overline{\Psi}\Psi = \frac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{i}{2}(\partial_\mu\overline{\Psi})\gamma^\mu\Psi - m\overline{\Psi}\Psi\,. \tag{28}$$

(a) $\mathcal{L}$ is invariant under the global transformation $\Psi \to e^{-i\alpha}\Psi$, $\overline{\Psi} \to e^{i\alpha}\overline{\Psi}$. Show that the associated Noether current is $j^\mu = \overline{\Psi}\gamma^\mu\Psi$.

(b) Using the Euler–Lagrange equations show explicitly that j^μ is conserved.

(c) We can define another current as $j^\mu_{\mathrm{A}} = \overline{\Psi}\gamma^\mu\gamma^5\Psi$. Compute the four-divergence of j^μ_{A}. When is this current conserved?

(d) Write down the energy momentum tensor! What is the Hamiltonian density? (Bonus: prove by explicit calculation that the energy momentum tensor is real.)

Problem 1.7* Relativistic motion of a charged particle

The equation of motion of a relativistic point particle of charge q in an electromagnetic field is

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu\,, \tag{29}$$

where τ is the proper time and

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{30}$$

is the four-velocity, related to the four-momentum by $p^\mu = mu^\mu$. Show that in coordinate time t the spatial part of this equation accounts for the Lorentz force and the time component captures the work of the field on the particle.

Problem 1.8 Scalar electrodynamics

Consider the Lagrangian density of the free complex scalar field

$$\mathcal{L} = \partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi. \tag{31}$$

(a) $\mathcal{L}$ is invariant under the global transformation

$$\Phi \to e^{-iq\alpha}\Phi\,, \qquad \Phi^* \to e^{iq\alpha}\Phi^*\,. \tag{32}$$

Show that the associated Noether current is

$$j^\mu = iq\left(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*\right)\,. \tag{33}$$

(b) Let us take now a *local* gauge transformation,

$$\Phi \to e^{-iq\alpha(x)}\Phi\,, \qquad \Phi^* \to e^{iq\alpha(x)}\Phi^*\,. \tag{34}$$

Show that the Lagrangian density of scalar electrodynamics is

$$\mathcal{L} = (D_\mu\Phi)^*(D^\mu\Phi) - m^2\Phi^*\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{35}$$

is invariant under these transformations if the covariant derivative is defined as

$$D_\mu = \partial_\mu + iqA_\mu \tag{36}$$

and we also transform A_μ as $A_\mu \to A_\mu + \partial_\mu\Lambda$.

(c) Show that the Euler–Lagrange equation obtained by varying with respect to Φ^* can be written as

$$(D_\mu D^\mu + m^2)\Phi = 0\,. \tag{37}$$

(d) Derive the Euler–Lagrange equation corresponding to the variation of A_μ and write it in terms of $F^{\mu\nu}$ and j^μ. Note the difference with respect to the equation you found in 1.4 c).

(e) Note that the current is not gauge invariant. Using its transformation property show that the equation of motion derived in d) is gauge invariant.

Problem 1.9 Yukawa theory

Yukawa proposed that the strong interaction is mediated by the π meson between protons and neutrons. The proton and the neutron are described by Dirac 4-spinors and (neglecting their mass difference) form a doublet under $SU(2)_I$ isospin symmetry. This can be formulated by the following Lagrangian:

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - M)\Psi + \frac{1}{2}\partial_\mu \pi^j \partial^\mu \pi^j - \frac{1}{2}m_\pi^2 \pi^j \pi^j - \frac{\lambda}{4}(\pi^j \pi^j)^2 + g\overline{\Psi}\gamma^5 \sigma^j \Psi \pi^j, \tag{38}$$

where $j = 1, 2, 3$ with summation over repeated indices, $\Psi = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}$ is the nucleon doublet composed of the Dirac spinors of the proton and neutron, and σ^j are the Pauli matrices.

(a) Derive the Euler–Lagrange equations.

(b) Obtain the Noether current corresponding to an $SU(2)_I$ symmetry transfromation.
Note: remember that we can get our familiar pions with $\pi^3 = \pi^0$, *and* $\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm \pi^2)$. *Also note that they form an isospin triplet:* $\pi^0 = |1, 0\rangle$ *and* $\pi^\pm = |1, \pm 1\rangle$.