1 The scattering cross section

1.1 Derivation of the differential cross section for a general $2 \to n$ particle scattering process

Let us write down the oncoming wave packets as

$$|i\rangle = \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)|\vec{k}_A, \vec{k}_B\rangle$$

and the outgoing wave packes as

$$|f\rangle = \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i) \right\} |\vec{p}_1, \dots, \vec{p}_n\rangle$$

The S matrix is written as

$$\mathbb{S} = 1 + i\mathbb{T}$$

and the transition amplitude can be computed easily

$$T(\vec{b}) = \langle f|\mathbb{T}|i\rangle = \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i)^* \right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B} \langle \vec{p}_1, \dots, \vec{p}_n|\mathbb{T}|\vec{k}_A, \vec{k}_B\rangle$$

where

$$\langle \vec{p}_1, \dots, \vec{p}_n|\mathbb{T}|\vec{k}_A, \vec{k}_B\rangle = -\mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \dots, \vec{p}_n\}\right)(2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)$$

and the factor $e^{-i\vec{b}\cdot\vec{k}_B}$ corresponds to an incoming projectile B with an impact parameter $\vec{b}$ relative to the target particle A. The transition probability for this process is

$$\begin{aligned}
P(\vec{b}) &= |T(\vec{b})|^2 \\
&= \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i)^* \right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B} \langle \vec{p}_1, \dots, \vec{p}_n|\mathbb{T}|\vec{k}_A, \vec{k}_B\rangle \\
&\quad \prod_{i=1}^{n} \left\{ \int \frac{d^3p'_i}{(2\pi)^3\sqrt{2\omega'_i}} \phi_i(\vec{p}'_i) \right\} \int \frac{d^3k'_A}{(2\pi)^3\sqrt{2\omega'_A}} \int \frac{d^3k'_B}{(2\pi)^3\sqrt{2\omega_B}'} \phi_A(\vec{k}'_A)^*\phi_B(\vec{k}'_B)^* e^{i\vec{b}\cdot\vec{k}'_B} \langle \vec{p}'_1, \dots, \vec{p}'_n|\mathbb{T}|\vec{k}'_A, \vec{k}'_B\rangle^* \\
&= \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i)^* \right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \prod_{i=1}^{n} \left\{ \int \frac{d^3p'_i}{(2\pi)^3\sqrt{2\omega'_i}} \phi_i(\vec{p}'_i) \right\} \int \frac{d^3k'_A}{(2\pi)^3\sqrt{2\omega_A}'} \int \frac{d^3k'_B}{(2\pi)^3\sqrt{2\omega'_B}} \\
&\quad \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}'_A)^*\phi_B(\vec{k}'_B)^* e^{i\vec{b}\cdot(\vec{k}'_B - \vec{k}_B)}(2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k'_A + k'_B - \sum_{i=1}^{n} p'_i\right) \\
&\quad \mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \dots, \vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}'_A, \vec{k}'_B\} \to \{\vec{p}'_1, \dots, \vec{p}'_n\}\right)^*
\end{aligned}$$

What is the number of events per second of the incident beam scattering on the target particle? If the beam has a particle flux j_B (particles per second per area) then the number of events per second (event frequency) is

$$f = \int d^2b j_B P(\vec{b}) = j_B \int d^2b P(\vec{b})$$

where I supposed that the incoming beam flux is homogeneous across it's transverse area so it can be brought out of the integration. Then the cross section is just

$$\sigma = \int d^2b P(\vec{b})$$

Now our first step is to suppose that the outgoing states are narrow wave packets so that $\phi_i(\vec{p}_i)^*\phi_i(\vec{p}'_i)$ is only nonzero when $\vec{p}_i \approx \vec{p}'_i$ (what this really means that our detectors measuring the final state have a good momentum resolution). Using the wave packet normalisation

$$\int \frac{d^3p}{(2\pi)^3} |\phi(\vec{p})|^2 = 1$$

we can perform the integral over the $\vec{p}_i'$ leaving us with

$$P(\vec{b}) = \prod_{i=1}^{n} \left\{ \int \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \int \frac{d^3 k_A}{(2\pi)^3 \sqrt{2\omega_A}} \int \frac{d^3 k_B}{(2\pi)^3 \sqrt{2\omega_B}} \int \frac{d^3 k_A'}{(2\pi)^3 \sqrt{2\omega_A}'} \int \frac{d^3 k_B'}{(2\pi)^3 \sqrt{2\omega_B'}}$$
$$\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}_A')^*\phi_B(\vec{k}_B')^* e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}(2\pi)^4\delta^{(4)}\left(k_A+k_B-\sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k_A'+k_B'-\sum_{i=1}^{n} p_i\right)$$
$$\mathcal{M}\left(\{\vec{k}_A,\vec{k}_B\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}_A',\vec{k}_B'\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)^*$$

Now we omit the final state integral to write the differential cross section as

$$d\sigma = \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \int d^2 b \int \frac{d^3 k_A}{(2\pi)^3 \sqrt{2\omega_A}} \int \frac{d^3 k_B}{(2\pi)^3 \sqrt{2\omega_B}} \int \frac{d^3 k_A'}{(2\pi)^3 \sqrt{2\omega_A}'} \int \frac{d^3 k_B'}{(2\pi)^3 \sqrt{2\omega_B'}}$$
$$\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}_A')^*\phi_B(\vec{k}_B')^* e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}(2\pi)^4\delta^{(4)}\left(k_A+k_B-\sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k_A'+k_B'-\sum_{i=1}^{n} p_i\right)$$
$$\mathcal{M}\left(\{\vec{k}_A,\vec{k}_B\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}_A',\vec{k}_B'\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)^*$$

We can now perform the integral over the impact parameter to get

$$d\sigma = \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \int \frac{d^3 k_A}{(2\pi)^3 \sqrt{2\omega_A}} \int \frac{d^3 k_B}{(2\pi)^3 \sqrt{2\omega_B}} \int \frac{d^3 k_A'}{(2\pi)^3 \sqrt{2\omega_A'}} \int \frac{d^3 k_B'}{(2\pi)^3 \sqrt{2\omega_B'}}$$
$$\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}_A')^*\phi_B(\vec{k}_B')^*(2\pi)^2\delta^{(2)}\left(\vec{k}_{B\perp}'-\vec{k}_{B\perp}\right)(2\pi)^4\delta^{(4)}\left(k_A+k_B-\sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k_A'+k_B'-\sum_{i=1}^{n} p_i\right)$$
$$\mathcal{M}\left(\{\vec{k}_A,\vec{k}_B\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}_A',\vec{k}_B'\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)^*$$

where $\perp$ denotes the components x, y perpendicular to the direction z of the collision. Now the δ-functions enforce the following identities between the A, B four vectors:

$$\vec{k}_A + \vec{k}_B = \vec{k}_A' + \vec{k}_B'$$
$$\vec{k}_{B\perp} = \vec{k}_{B\perp}'$$
$$\sqrt{\vec{k}_A^2 + m_A^2} + \sqrt{\vec{k}_B^2 + m_B^2} = \sqrt{\vec{k}_A'^2 + m_A^2} + \sqrt{\vec{k}_B'^2 + m_B^2}$$

The first two equalities together enforce

$$\vec{k}_{A\perp} = \vec{k}_{A\perp}' \qquad \vec{k}_{B\perp} = \vec{k}_{B\perp}'$$

while the equations for the z components are

$$k_A^z + k_B^z = k_A'^z + k_B'^z$$
$$\omega_A + \omega_B = \sqrt{\left(k_A'^z\right)2 + \vec{k}_{A\perp}^2 + m_A^2} + \sqrt{\left(k_B'^z\right)2 + \vec{k}_{B\perp}^2 + m_B^2}$$

which enforces

$$k_A'^z = k_A^z \qquad k_B'^z = k_B^z$$

So the result of the k' integration is

$$\int \frac{d^3 k_A'}{(2\pi)^3 \sqrt{2\omega_A'}} \int \frac{d^3 k_B'}{(2\pi)^3 \sqrt{2\omega_B'}} (2\pi)^2\delta^{(2)}\left(\vec{k}_{B\perp}'-\vec{k}_{B\perp}\right)(2\pi)^4\delta^{(4)}\left(k_A'+k_B'-k_A-k_B\right)(\ldots)$$
$$= \frac{1}{\sqrt{2\omega_A}}\frac{1}{\sqrt{2\omega_B}}\int dk_A'^z\ \delta\left(\sqrt{\left(k_A'^z\right)^2+\vec{k}_{A\perp}^2+m_A^2}+\sqrt{\left(k_B'^z\right)^2+\vec{k}_{B\perp}^2+m_B^2}-\omega_A-\omega_B\right)\Bigg|_{k_B'^z=k_A^z+k_B^z-k_A'^z}$$
$$(\ldots)|_{\vec{k}_{A,B}=\vec{k}_{A,B}'}$$
$$= \frac{1}{\sqrt{2\omega_A}}\frac{1}{\sqrt{2\omega_B}}\frac{1}{\left|\frac{k_A^z}{\omega_A}-\frac{k_B^z}{\omega_B}\right|}\ (\ldots)|_{\vec{k}_{A,B}=\vec{k}_{A,B}'}$$

where we used the rule

$$\delta(f(x)) = \frac{1}{|f'(a)|}\delta(x-a)$$

when the function $f(x)$ has a single zero at $x=a$, and the result

$$\frac{\partial}{\partial k_A'^z}\left(\sqrt{\left(k_A'^z\right)^2+\vec{k}_{A\perp}^2+m_A^2}+\sqrt{\left(k_B'^z\right)^2+\vec{k}_{B\perp}^2+m_B^2}-\omega_A-\omega_B\right)\Bigg|_{k_B'^z=k_A^z+k_B^z-k_A'^z}$$
$$=\frac{k_A'^z}{\sqrt{\left(k_A'^z\right)^2+\vec{k}_{A\perp}^2+m_A^2}}-\frac{k_B'^z}{\sqrt{\left(k_B'^z\right)^2+\vec{k}_{B\perp}^2+m_B^2}}$$

which is just equal to

$$\frac{k_A^z}{\omega_A}-\frac{k_B^z}{\omega_B}$$

once we use that $\vec{k}_A=\vec{k}_A'$ and $\vec{k}_B=\vec{k}_B'$. Now note that

$$\frac{k_A^z}{\omega_A}-\frac{k_B^z}{\omega_B}=v_A-v_B$$

where v_A and v_B are the velocities of the particles A and B in the direction of the scattering axis z.

So the result is

$$d\sigma=\prod_{i=1}^n\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\}\int\frac{d^3k_A}{(2\pi)^3 2\omega_A}\int\frac{d^3k_B}{(2\pi)^3 2\omega_B}\frac{1}{|v_A-v_B|}\left|\phi_A(\vec{k}_A)\right|^2\left|\phi_B(\vec{k}_B)\right|^2$$
$$(2\pi)^4\delta^{(4)}\left(k_A+k_B-\sum_{i=1}^n p_i\right)\left|\mathcal{M}\left(\{\vec{k}_A,\vec{k}_B\}\to\{\vec{p}_1,\dots,\vec{p}_n\}\right)\right|^2$$

Now we assume that the wave packets $\phi_{A,B}$ are very narrow (i.e. the incoming particles have a sharply defined value of momentum) and that the amplitude $\mathcal{M}$ varies slowly enough so that it can be taken constant on their support. Then we can write

$$d\sigma=\prod_{i=1}^n\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\}\frac{1}{2\omega_A}\frac{1}{2\omega_B}\frac{1}{|v_A-v_B|}(2\pi)^4\delta^{(4)}\left(k_A+k_B-\sum_{i=1}^n p_i\right)\left|\mathcal{M}\left(\{\vec{k}_A,\vec{k}_B\}\to\{\vec{p}_1,\dots,\vec{p}_n\}\right)\right|^2$$

which is our main result.

1.2 Lorentz properties of the cross section

All the ingredients of $d\sigma$ are Lorentz invariant with the exception of the factor

$$\frac{1}{\omega_A\omega_B|v_A-v_B|}=\frac{1}{|k_A^z\omega_B-k_B^z\omega_A|}$$

This should transform as area element in the xy plane. Let us check this explicitly. Lorentz transformations consist of rotations and boosts in three principal directions x, y and z. Let us go through each of these one by one.

1. Rotations around z axis do not change neither the z component of momentum, nor the energy, so this is invariant. This is right since rotation leaves an area element perpendicular to the axis invariant.

2. Boosts in direction z with velocity u (in units of $c=1$) result in

$$\omega'=\gamma(\omega-uk^z)$$
$$k'^z=\gamma(k^z-u\omega)\qquad\gamma=(1-u^2)^{-1/2}$$

   So we get

$$|k_A^z\omega_B-k_B^z\omega_A|\to\gamma^2\left|(k_A^z-u\omega_A)(\omega_B-uk_B^z)-(k_B^z-u\omega_B)(\omega_A-uk_A^z)\right|=|k_A^z\omega_B-k_B^z\omega_A|$$

   which is correct as area elements perpendicular to the boost direction do not undergo Lorentz contraction.

3. Boosts in direction x with velocity u (in units of $c = 1$) result in

$$\begin{aligned} \omega' &= \gamma(\omega - uk^x) \\ k'^x &= \gamma(k^x - u\omega) \qquad \gamma = (1-u^2)^{-1/2} \end{aligned}$$

   Now since the particles move in direction z, k^x is zero, so we get

$$|k_A^z \omega_B - k_B^z \omega_A| \to \gamma\, |k_A^z \omega_B - k_B^z \omega_A|$$

   so

$$d\sigma \to \frac{1}{\gamma} d\sigma = \sqrt{1-u^2} d\sigma$$

   This is just the result of Lorentz contraction in direction x.
   The calculation for a boost in direction y is similar, and leads to the same result.

4. Rotation around $x$or y axis does not change the energy, but changes the direction of the momentum. Since the original x and y components are zero, the z components of the momentum picks up a factor $\cos\alpha$ where α is the rotation angle, so we get

$$d\sigma \to \frac{d\sigma}{\cos\alpha}$$

   This is exactly the correct way for the transformation of a cross sectional area element under rotation.

1.3 $2 \to 2$ processes

For the special case of a two-particle scattering process of two particles of masses m_A and m_B to two other particles with masses m_1 and m_2 we have the conservation equations

$$\begin{aligned} \vec{k}_A + \vec{k}_B &= \vec{p}_1 + \vec{p}_2 \\ E_A + E_B &= E_1 + E_2 \end{aligned}$$

and the differential cross section is

$$d\sigma = \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} (2\pi)^4 \delta^{(4)} \left(k_A + k_B - p_1 - p_2\right) \left|\mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \vec{p}_2\}\right)\right|^2$$

Note that we have 6 differentials, but also 4 Dirac deltas, which means that we can fix four of the differential variables (i.e. integrate them out). It is simplest to do this in the centre-of-mass (or zero-momentum frame, in which $\vec{k}_A + \vec{k}_B = 0$ which means that the spatial part of $\delta^{(4)}$ enforces $\vec{p}_2 = -\vec{p}_1$. This allows us to integrate over $\vec{p}_2$ to obtain

$$d\sigma = \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{1}{2E_2} \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} 2\pi\delta\left(E_{CM} - E_1 - E_2\right) |\mathcal{M}|^2$$

where

$$E_{CM} = E_A + E_B$$

is the centre-of-mass total energy. Now the last δ fixes $p_1 = |\vec{p}_1|$ since it means that

$$E_{CM} - \sqrt{p_1^2 + m_1^2} - \sqrt{p_1^2 + m_2^2} = 0$$

To use this we write $d^3p_1 = p_1^2 dp_1 d\Omega$ where Ω is the solid angle parametrizing of the direction of the outgoing particle A. The derivative of the Dirac delta argument with respect to p_1 is

$$-\left(\frac{p_1}{E_1} + \frac{p_1}{E_2}\right)$$

so after integration we are left by

$$\begin{aligned} d\sigma &= \frac{p_1^2 d\Omega}{4\pi^2} \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} \frac{1}{\frac{p_1}{E_1} + \frac{p_1}{E_2}} |\mathcal{M}|^2 \\ &= d\Omega \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} \frac{|\vec{p}_1|}{16\pi^2(E_1 + E_2)} \end{aligned}$$

What is all the masses are equal $m_A = m_B = m_1 = m_2 = m$. This leads to $\left|\vec{k}_A\right| = \left|\vec{k}_B\right| = |\vec{p}_1| = |\vec{p}_2|$ and so $E_A = E_B = E_1 = E_2 = E_{CM}/2$. In addition

$$v_A - v_B = \frac{k_A^z}{E_A} - \frac{k_B^z}{E_B} = 2\frac{k_A^z}{E_A} = 2\frac{|\vec{p}_1|}{E_{CM}/2} = 4\frac{|\vec{p}_1|}{E_{CM}}$$

so we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2}$$

In our scalar example we had $\mathcal{M} = \lambda + O(\lambda^2)$, so the differential cross section to first order is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{64\pi^2 E_{CM}^2}$$

Note that

$$E_{CM}^2 = (p_1 + p_2)^2$$

where the square denotes the Lorentz "norm". The Mandelstam variable

$$s = (p_1 + p_2)^2$$

provides a Lorentz-invariant expression for the centre-of-mass energy (squared).

The total cross section can be obtained by

$$\sigma = \frac{1}{2}\int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{32\pi s}$$

where the factor 1/2 takes into account that there are two identical particles in the end state and we cannot distinguish which one was emitted in the given angular direction over which we integrate.

2 The decay rate

If there is only a single particle in the initial state, the process describes the decay. As discussed in the class, the proper way of obtaining the decay rate Γ is via the construction of the relativistic Breit-Wigner distribution of the unstable (resonant) excitation, but the end result can be easily guessed modifying the formula for $d\sigma$ by eliminating one of the incoming particles. The result is

$$d\Gamma = \prod_{i=1}^{n}\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\}\frac{1}{2\omega_A}(2\pi)^4\delta^{(4)}\left(k_A - \sum_{i=1}^{n} p_i\right)\left|\mathcal{M}\left(\{\vec{k}_A\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\right|^2$$

The total decay rate can be obtained by integrating over all the final states. Note that the decay rate Γ_0 in the rest frame is obtained by substituting ω_A by the mass m_A, therefore

$$\Gamma = \frac{m_A}{\omega_A}\Gamma_0$$

Since the decay rate is related to the lifetime $\tau$by $\Gamma = 1/\tau$, we obtain

$$\tau = \frac{\omega_A}{m_A}\tau_0$$

where

$$\frac{\omega_A}{m_A} = \frac{1}{\sqrt{1 - u_A^2}}$$

with u_A being the velocity of particle A in $c = 1$ units. This is just the correct formula for relativistic time dilation, thus supporting our guess. In fact, the formula for the decay rate can be proven rigorously using the so-called optical theorem (cf. Section 7.3 in Peskin-Schroeder).

3 Some dimensional analysis

The dimensions (in units of energy) of a momentum Dirac delta can be obtained from

$$\int d^3p\delta^{(3)}(\vec{p}) = 1$$

Since $\left[d^3p\right] = 3$ we get $\left[\delta^{(3)}(\vec{p})\right] = -3$. Now our states have inner products

$$\langle \vec{p}' | \vec{p} \rangle = 2\omega_{\vec{p}}\delta^{(3)}(\vec{p} - \vec{p}')$$

The left hand side has dimension -2, so a one-particle state has dimension -1. Therefore, the dimensionality of a multi-particle state is given by

$$[|\vec{p}_1, \ldots, \vec{p}_n\rangle] = -n$$

Now the S-matrix is dimensionless since it is a (time-ordered) exponential of the time-integrated Hamiltonian density, and so is $\mathbb{T}$. Therefore, using

$$\langle \vec{p}_1, \ldots, \vec{p}_n | \mathbb{T} | \vec{k}_1, \ldots, \vec{k}_m \rangle = -\mathcal{M}\left(\{\vec{k}_1, \ldots, \vec{k}_m\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)(2\pi)^4\delta^{(4)}\left(\sum_{j=1}^{m}\vec{k}_j - \sum_{i=1}^{n}p_i\right)$$

we obtain that

$$\left[\mathcal{M}\left(\{\vec{k}_1, \ldots, \vec{k}_m\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\right] = -n - m + 4$$

Let us check the dimensionality of the cross section

$$d\sigma = \prod_{i=1}^{n}\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\}\frac{1}{2\omega_A}\frac{1}{2\omega_B}\frac{1}{|v_A - v_B|}(2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n}p_i\right)\left|\mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\right|^2$$

We obtaind that this is

$$[d\sigma] = n(3-1) - 2 - 4 + 2(4 - n - 2) = -2$$

which is just right for an area.

For the decay rate

$$d\Gamma = \prod_{i=1}^{n}\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\}\frac{1}{2\omega_A}(2\pi)^4\delta^{(4)}\left(k_A - \sum_{i=1}^{n}p_i\right)\left|\mathcal{M}\left(\{\vec{k}_A\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\right|^2$$

we get

$$[d\Gamma] = n(3-1) - 1 - 4 + 2(4 - n - 1) = 1$$

which is correct since it must have the same units as energy (being inversely proportional to time).