Problem set for the course on one-dimensional electron systems

Rules: You can select problems from each set to collect 50 points at maximum (i.e. max. 100 points). To pass (grade 2) you need to reach 50%, for grade 3 you need 60%, for grade 4 you need to reach 70%, and for grade 5, you must score above 80%. You may discuss with the others (or with me), give hints to each-other, but I request independent work. You may help each-other but you are not allowed to copy For those, who do not meet the deadline: the total points they get decreases by 5 each day after the deadline. (Thus a two-day delay implies a loss of 10 points).
You can get extra points by solving optional parts of a problem.

SET 1

Symmetries of the Hubbard model (30 pts)
The Half-filled Hubbard model is defined as

$$H = -t\sum_{i,\sigma}(c^{\dagger}_{i,\sigma}c_{i+1,\sigma} + h.c.) + U\sum_{i}(n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2)\ .$$

Define the following 'pseudospin' operators:

$$J^z = \frac{1}{2}\sum_i(c^{\dagger}_{i\uparrow}c_{i\uparrow} + c^{\dagger}_{i\downarrow}c_{i\downarrow} - 1)\ , \qquad J^+ = \sum_i(-1)^i c^{\dagger}_{i\downarrow}c^{\dagger}_{i\uparrow} \qquad J^- = \sum_i(-1)^i c_{i\uparrow}c_{i\downarrow}\ , \tag{1}$$

and the spin operators,

$$S^z = \frac{1}{2}\sum_i(c^{\dagger}_{i\uparrow}c_{i\uparrow} - c^{\dagger}_{i\downarrow}c_{i\downarrow})\ , \qquad S^+ = \sum_i c^{\dagger}_{i\uparrow}c_{i\downarrow}\ , \qquad S^- = \sum_i c^{\dagger}_{i\downarrow}c_{i\uparrow}\ . \tag{2}$$

a. (15pts) Show that these operators satisfy the usual SU(2) commutation relations, and that they commute with each other. Show also that the commute with H. (This implies that the J's are generators of a hidden SU(2) symmetry. The corresponding Goldstone-modes are responsible for the destruction of true antiferromagnetic and superconducting order at zero temperature.)

b. (15pts) Discuss the spectrum of a two-site problem, without actually solving for the eigenvalues. (Hint: Consider the eigenstates of the single site model, as classified by symmetries, and then use spin addition rules to show that there are 2 4-fold degenerate multiplets, 2 3-fold degenerate states, and 2 non-degenerate states. Determine their quantum numbers! Each state can be further classified by *parity*. Which state, do you expect to be the ground state for $U > 0$ and for $U < 0$? Construct these states explicitly!

Spin coherent states (30+5 points)
In this problem, you will use Schwinger bosons to construct spin coherent states with $S > 1$. The basic idea of Schwinger bosons is simply to have bosons 'a', corresponding to spin $\uparrow$ and 'b', corresponding to spin $\downarrow$. Taking N spin one-half bosons one gets a spin S=N/2 state. This is mathematically formulated in the following way.

a. (5pts) Show that the operators

$$S^+ \equiv a^\dagger b, \qquad S^- \equiv b^\dagger a, \qquad S^z = (a^\dagger a - b^\dagger b)/2 \tag{3}$$

represent the usual spin S operators within the subspace $n_a + n_b = a^\dagger a + b^\dagger b = 2S$, i.e., (1) they satisfy the usual SU(2) commutation relations, and (2) their squares add up to S(S+1). Show also, that the states

$$|S, m\rangle \equiv \frac{(a^\dagger)^{S+m}(b^\dagger)^{S-m}}{\sqrt{(S+m)!(S-m)!}}|0\rangle$$

are the usual, well normalized eigenstates of S_z.

b. (10pts) Now construct the rotation operator at class, and show that the operator rotating the z axis into axis $\Omega = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$ along a "straight line" can be written as

$$R(\theta, \varphi) = \exp\{zS^+ - \bar{z}S^-)$$

with $z = -\theta\, e^{-i\varphi}/2$. (Hint: rotate around the axis $(-\sin(\varphi), \cos(\varphi), 0)$ by angle θ.) Show now that

$$R(\theta, \varphi)\ (a^\dagger, b^\dagger)\ R^\dagger(\theta, \varphi) = (a^\dagger, b^\dagger) \cdot U(\theta, \varphi),$$

with $U = \exp\{z\sigma^+ - \bar{z}\sigma^-)$ a 2x2 spin 1/2 rotation matrix. Compute this latter explicitly!

c. (10pts) Show now by explicit calculation that

$$|\Omega\rangle \equiv R(\theta, \varphi)|z\rangle = \sum_{m=-S}^{S} \sqrt{\binom{2S}{m+S}}\, (\cos(\theta/2))^{S+m} \left(\sin(\theta/2)e^{i\varphi}\right)^{S-m} |m\rangle$$

Hint: represent the state $|m = S\rangle$ in terms of bosons, and act on it with R. Use that the vacuum is invariant under rotations, $R|0\rangle = |0\rangle$. Then you can express the spin coherent state in terms of the rotated boson operator, $\tilde{a}^\dagger = Ra^\dagger R^\dagger$. Use the result of point b to express now this as a linear combination of $a^\dagger$ and $b^\dagger$, and then use the binomial theorem to obtain the expansion above.

d. (5pts) Now define the coherent state wave function as $\psi(\omega) \equiv \langle\Omega|m\rangle$. Plot $|\psi(\omega)|^2$ for $S = 1$ and $m = \pm 1$ and 0. Show that $|\psi(\omega)|^2$ integrates to $4\pi/(2S+1) = 4\pi/3$ over the sphere in all three cases.

e. For fun (or +5 pts): Prove the (over)completeness relation from c.

$$\int \frac{d^2\Omega}{2S+1} |\Omega\rangle\langle\Omega| = 1,$$

and compute the overlap of two different coherent states!

SET 2

(30p) Verification of renormalization group equations
In this problem, we consider an interacting electron gas in the presence of forward and backward scattering, g_2 and g_1, and umklapp processes, g_3. Your task is to generate the RG equations for the coupling g_2.

a. (10pts) First compute *all* second order vertex corrections to γ_2. Take external momenta to 0, and keep only one frequency, for simplicity. Show that only four diagrams remain, two with crossed interaction lines, and two ladder type diagrams.

b. (10pts) Now add up these contributions, and obtain the renormalized coupling $\tilde{g}_2$ by rescaling the cut-off $\Lambda \to \tilde{\Lambda}$ and compensating that be a renormalization $g_2 \to \tilde{g}_2$. Derive the RG equations

$$\frac{\mathrm{d}g_2}{\mathrm{d}l} = g_3^2 - g_1^2 \; .$$

c. (10pts) Assume now that $g_4 \neq 0$. Draw all diagrams which could then contribute to γ_2, and show that they are not logarithmically singular, i.e., they scale to 0 as $\omega \to 0$.

Some details of the Bethe Ansatz (BA) solution of the Heisenberg model (30)
In this problem, you should work out some of the details of the BA solution of the Heisenberg chain,

$$H = \sum_{i=1}^{L} \Big(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta \; S_i^z S_{i+1}^z \Big) \; . \tag{4}$$

Within the BA, one writes the eigenstates in the following form:

$$|\psi\rangle = \sum_{1 \le x_1 < x_2 < \cdots < x_N \le L} f(\{x_i\}) \; |\{x_i\}\rangle \; , \tag{5}$$

with $f(\{x_i\})$ the BA wave function, and $\{x_i\}$ the positions of the spins pointing downwards.

a. (10p) First show that for just two particles the wave function can be written as

$$f(x_1, x_2) = A_{12} e^{i(k_1 x_1 + k_2 x_2)} + A_{21} e^{i(k_1 x_2 + k_2 x_1)} \; , \tag{6}$$

and that proper matching of the boundary conditions leads to

$$A_{21} = -\frac{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_2}}{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_1}} \; A_{12} \equiv S(k_1, k_2) \; A_{12} \; . \qquad (3p)$$

Discuss in detail the quantization conditions in case of periodic boundary conditions, and display explicitly the corresponding equations for k_1 and k_2 (6p). Show that $k_1 + k_2 = (2\pi/L)\, n$, with n some integer (1p).

b. (10p) Introduce now the rapidity variable λ through

$$e^{ik} = \frac{2\lambda - i}{2\lambda + i} \; .$$

Show that for $\Delta = 1$ one can express the S-matrix obtained in (a) as (2p)

$$S(k_1, k_2) = \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \; ,$$

and that the energy (2p) and the periodic boundary conditions (2p) read:

$$E = \frac{L}{4} - \sum_i \frac{2}{4\lambda_i^2 + 1} \ , \tag{7}$$

$$\left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^L = \prod_{k=1}^{N} \left(\frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}\right) \ . \tag{8}$$

Take the logarithm of the last equation and derive the quantization conditions (4p):

$$L \ 2\arctan(2\lambda_j) = \sum_{k=1}^{N} 2\arctan(\lambda_j - \lambda_k) + 2\pi \ J_j \ ,$$

with J_j an integer/half-integer.

c. **(10 pts)** Assuming that in the ground state the J_j's are just $N = L/2$ consecutive integers (or half-integers), and thus there are no holes, introduce the ground state density of rapidities, $\sigma_0(\lambda)$, and derive the self-consistency equation (2p):

$$\sigma_0(\lambda) = \frac{2L}{\pi} \frac{1}{4\lambda^2 + 1} - \frac{1}{\pi} \int \mathrm{d}\lambda' \ \frac{2}{(\lambda - \lambda')^2 + 1} \sigma_0(\lambda') \ . \tag{9}$$

Take the Fourier transform of this equation, and show that (4p)

$$\sigma_0(\omega) = \frac{L}{2} \frac{1}{\cosh(\omega/2)} \ .$$

Now express the ground state energy in terms of $\sigma_0(\lambda)$, and show that (4p)

$$E_0 = \frac{L}{4}(1 - 4\ln 2) \ .$$

Hint: you do not need to compute $\sigma_0(\lambda)$ explicitly, the integral of the product of two functions can be evaluated in Fourier space.

Spin-charge separation in an interacting electron gas (25pts)
Consider the fixed point Hamiltonian of a spinful interacting electron gas

$$H = H_0 + 2\pi v_F g_4 \int \left(\mathrm{d}x \ :R_\uparrow^\dagger R_\uparrow R_\downarrow^\dagger R_\downarrow : + \text{"} L \leftrightarrow R \text{"}\right) + 2\pi v_F g_2 \sum_{\sigma,\sigma'} \int \mathrm{d}x \ :R_\sigma^\dagger R_\sigma L_{\sigma'}^\dagger L_{\sigma'} : \ .$$

a. (5pts) Introduce the right- and left-mover bosonic fields, $\Phi_{\sigma,\pm}$, similarly to what we did at class, and express in terms of these the density-density interaction terms and the non-interacting part of the Hamiltonian (neglect Klein factors).

b. (10pts) Next, introduce the 'spin' and 'charge' fields,

$$\phi_{s,\pm} \equiv \frac{1}{\sqrt{2}}(\phi_{\uparrow,\pm} - \phi_{\uparrow,\pm}) \ , \qquad \phi_{c,\pm} \equiv \frac{1}{\sqrt{2}}(\phi_{\uparrow,\pm} + \phi_{\uparrow,\pm}),$$

and the corresponding fields $\varphi_s = (\phi_{s,+} + \phi_{s,-})/\sqrt{4\pi}$ and $\vartheta_s = (\partial_x\phi_{s,-} - \partial_x\phi_{s,+})/\sqrt{4\pi}$, and similarly the charge fields. Verify that the charge and spin fields φ and ϑ satisfy the canonical commutation relations, and they anticommute with each other.

c. (10pts) Now rewrite the interacting Luttinger liquid Hamiltonian in terms of the fields, $\varphi_{s,c}$ and $\vartheta_{s,c}$, and show that charge ans spin degrees of freedom decouple. Using the commutation relations, generate the equations of motions for $\vartheta_{s,c}$ and $\varphi_{s,c}$, and show that they satisfy the standard wave equation,

$$(\partial_t^2 - u_{s,c}^2 \partial_x^2)\, \varphi_{s,c} = 0,$$

but with different spin and charge propagation velocities, u_c and u_s. Determine these velocities in terms of v_F and the couplings g_2 and g_4. Determine also the Luttinger parameters for the spin and charge sector!

(20pts) Bosonization for left-movers

Repeat all calculations done for right-movers also for left-movers. Construct the field

$$\phi_L(x) \equiv \sum_{q>0} \frac{i}{\sqrt{n_q}} \big(b_{L,q}\, e^{-i\,qx} - b_{L,q}^\dagger\, e^{i\,qx}\big) e^{-aq/2} = \varphi_L + \varphi_L^\dagger\ . \tag{10}$$

a. (5 pts) Establish the connection between ϕ_L and the density of left-movers.

b. (5 pts) Compute the commutators $[\phi_L(x), \phi_L(x')]$ and $[\varphi_L(x), \varphi_L^\dagger(x')]$.

c. (5 pts) Show that $e^{-i\phi_L(x)}$ acts as an annihilation operator by computing its commutator with $\partial_x \phi_L$.

d. (5 pts) Express the non-interacting Hamiltonian in terms of this field.