MATHEMATICAL PHYSICS

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Chapter 1

COMPLEX VARIABLES

1.1 Introduction

A complex number is defined as a number z of the form $z = a + ib$, where $i = \sqrt{-1}$ and a and b are real numbers. a is called the real part of z [written as Re (z)] and b the imaginary part of z [written as Im (z)]. Thus, both Re (z) and Im (z) are real numbers. A complex number with the real part equal to zero is called a pure imaginary number. Operations involving complex numbers, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ can be defined as follows:

Addition and Subtraction: $\quad z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication: $\quad z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$$= a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2$$

since, $i^2 = -1$, we have $\quad z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

Division: $\quad \dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \dfrac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

Alternative definition

Complex numbers z can also be defined as ordered pairs of real numbers a and b, $z = (a, b)$, with the following rules for their addition and multiplication:

Addition:

$$z_1 = (a_1, b_1), \qquad z_2 = (a_2, b_2)$$

$$z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

Multiplication:

$$z_1 z_2 = (a_1, b_1)(a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$$

a is called the real part of the complex number $z = (a, b)$ and b is called the imaginary part. The complex number $(a, 0)$ is to be identified with the real number a. Complex numbers of the form $(0, b)$ are called pure imaginary numbers. $z = (a, b)$ can be written as

$$z = (a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0)$$

If the pure imaginary number (0, 1) is denoted by i (note that (0, 1) (0, 1) = (−1, 0) so that $i^2 = -1$), the ordered pair $z = (a, b)$ can be written $z = a + ib$.

A complex number is equal to zero if and only if its real and imaginary parts are both zero, i.e., $z = a + ib = 0$ implies $a = 0$, $b = 0$. This leads to the result that if two complex numbers are equal their real and imaginary parts are separately equal. Thus, if $z_1 = z_2$, $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2) = 0$ and therefore $a_1 = a_2$ and $b_1 = b_2$.

The complex number $z^* = a - ib$ is called the complex conjugate[1] of z. Evidently $(z^*)^* = z$ and $zz^* = (a + ib)(a - ib) = a^2 + b^2$ is real. $|z| \equiv \sqrt{a^2 + b^2}$ is called the modulous of z.

By putting $a = r\cos\theta$ and $b = r\sin\theta$, the complex number z can be written as $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. This is known as the polar form of z. $r = \sqrt{a^2 + b^2}$ is the modulous of z. θ is called argument of z (written arg z) and can be determined from the relation $\tan\theta = \dfrac{b}{a}$. Since $a \leqslant \sqrt{a^2 + b^2}$, it follows that $\text{Re}\,(z) \leqslant |z|$. Similarly $\text{Im}\,(z) \leqslant |z|$.

The complex number $z = a + ib$ can be represented by a point (a, b) in the xy-plane, the x-coordinate representing the real and the y-coordinate representing the imaginary part. This plane is called the complex plane or the z-plane (Fig. 1.1). $|z| = \sqrt{a^2 + b^2}$ is then the distance of the point from the origin and $\theta = \tan^{-1}\dfrac{b}{a} \pm 2n\pi$, $(n = 0, 1, 2, \ldots)$, is the angle the radius vector to the point makes with the x-axis. Thus the argument of z is not unique. It is conventional to restrict θ by the condition $-\pi < \theta \leqslant \pi$ and call it the principal value of arg z. The following identities involving moduli and arguments of complex numbers can be easily verified;

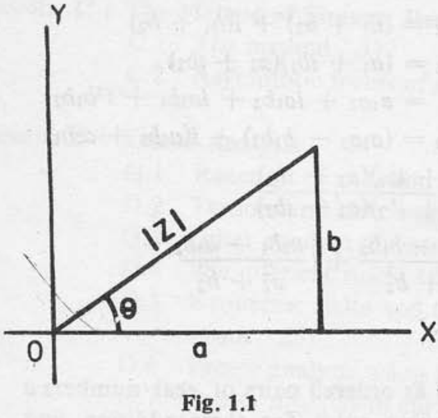

Fig. 1.1

$$|z_1 z_2| = |z_1| \cdot |z_2| \tag{1.1}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{1.2}$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \tag{1.3}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \tag{1.4}$$

The easiest way to verify them is by writing the complex numbers z_1 and z_2 in the polar form. Thus, if

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}$$

then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

[1]Some anthors use $\bar{z}$ instead of z^* to denote the complex conjugate of z.

This immediately gives $|z_1z_2| = r_1r_2 = |z_1| \cdot |z_2|$ and $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$. The other identities follow in a similar manner.

The following inequalities are extremely useful:

$$|z_1 + z_2| \leqslant |z_1| + |z_2| \tag{1.5}$$

$$|z_1 + z_2| \geqslant ||z_1| - |z_2|| \tag{1.6}$$

(1.5) is known as the *triangular inequality*. This can be proved as follows:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1 + z_2)^* \\ &= |z_1|^2 + |z_2|^2 + (z_1\overset{*}{z_2} + \overset{*}{z_1}z_2) \\ &= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\,(z_1\overset{*}{z_2}) \end{aligned}$$

But, $\quad \mathrm{Re}\,(z_1\overset{*}{z_2}) \leqslant |z_1\overset{*}{z_2}| = |z_1| \cdot |\overset{*}{z_2}| = |z_1| \cdot |z_2|$

Therefore, $\quad |z_1 + z_2|^2 \leqslant |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2|$

$$\leqslant (|z_1| + |z_2|)^2$$

or, $\quad |z_1 + z_2| \leqslant |z_1| + |z_2|$

To prove (1.6), we use the triangular inequality. If $|z_1| \geqslant |z_2|$,

$$|z_1| = |(z_1 + z_2) + (-z_2)| \leqslant |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$$

or, $$|z_1| - |z_2| \leqslant |z_1 + z_2| \tag{1.7}$$

On the other hand, if $\quad |z_1| < |z_2|$

$$|z_2| = |(z_1 + z_2) + (-z_1)| \leqslant |z_1 + z_2| + |z_1|$$

or $$|z_2| - |z_1| \leqslant |z_1 + z_2| \tag{1.8}$$

(1.7) and (1.8) together are equivalent to (1.6).

The result of measurement of a physical quantity is always a real number. One may then wonder what is the need for introducing complex numbers in physics. The reason is that the theory of functions of complex variables provide us with many powerful tools for calculation. These tools can be used to advantage if one introduces variables which are complex. A physical quantity will ultimately be related to either the real or imaginary part or the modulous-square of a complex variable.

1.2 Functions of a complex variable

For a set of values of z, the function $f(z)$ is a prescription for assigning for each z a value for $f(z)$. The set of values of z for which the function is defined is called the domain of z.

As for example let us consider the functions, (1) $f(z) = z^2$, (2) $f(z) = |z|^2$ and, (3) $f(z) = \ln z$, where $z = x + iy$.

(1) $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$,

Denoting the real and imaginary parts of the function by $u(x, y)$ and $v(x, y)$ respectively, we have $f(z) = u(x, y) + iv(x, y)$ where $u = x^2 - y^2$ and $v = 2xy$

(2) $f(z) = |z|^2 = x^2 + y^2$

In this case $u = x^2 + y^2$ and $v = 0$. Unlike the function in the first example which is complex, this function is real valued. The function of a complex variable can be complex, real or pure imaginary.

In the first two examples the domain of z is the set $-\infty < x < +\infty$ and $-\infty < y < +\infty$, i.e., the domain is the entire z-plane.

(3) In the third example, if we write $z = re^{i\theta}$, $f(z) = \ln z = \ln r + i\theta$. But z can also be written as $z = re^{i(2\pi n+\theta)}$, where in is an integer. Then $f(z) = \ln r + i(2\pi n + \theta)$. The peculiarity of this function is that it is not uniquely defined unless n is specified. Another way of stating the same thing is to say that the function is *multiple valued.* However, we would like to have, corresponding to every z, a *unique* assignment of value to $f(z)$. It is possible to have this for the function $\ln z$ by restricting the domain of z such that $n = 0$ and $-\pi < \theta \leqslant \pi$. The function defined with this restriction is known as the principal value of $\ln z$.

Similar difficulties arise with the function $f(z) = z^c$, where c is complex. $z^c = \exp(c \ln z)$ and because $\ln z$ is multiple valued z^c is not uniquely defined. The same ambiguity of definition is there, if c is real but not equal to an integer. If, however, $c = m =$ a real integer, then

$$\begin{aligned} z^m = \exp(m \ln z) &= \exp(m \ln r + im(2\pi n + \theta)) \\ &= r^m \exp[i(2\pi mn + m\theta)] \\ &= r^m \exp(im\,\theta) \end{aligned}$$

and z^m is uniquely defined. The ambiguities in the definition of z^c can be removed by restricting the domain of arg (z).

As in the case of functions of a real variable, we can define limit, continuity and derivative of a function of a complex variable.

Definition

w_0 is said to be the limit of $f(z)$ as z approaches z_0, if for each positive number ϵ there is a positive number δ such that $|f(z) - w_0| < \epsilon$, whenever $|z - z_0| < \delta$.

Intuitively this means that $f(z)$ can be brought arbitrarily close to w_0 by bringing z close to z_0, closeness between two complex numbers being measured by the modulous of their difference. This definition of the limit of $f(z)$ as $z \to z_0$ does not make any reference to the function at $z = z_0$

Definition

The function $f(z)$ is said to be continuous at z_0 if,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Definition

The derivative $f'(z_0)$ of a function $f(z)$ at z_0 is defined as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1.9}$$

Obviously, z_0 must be inside the domain of definition of $f(z)$.

If $f'(z_0)$ exists, the function $f(z)$ is said to be differentiable at z_0. The condition for the existence of $f'(z_0)$ implies that if the point z_0 in the z-plane is approached from any direction, the limit will be the same. In case of real variables, a point x_0 can be approached from only two directions, either from the left or from the right of x_0. The condition for the existence of derivatives of functions of complex variables is thus more stringent. This makes a lot of functions non-differentiable in complex variable theory.

Example 1. Consider the simple function $f(z) = z^*$. If we evaluate the limit $\lim\limits_{\Delta z\to 0} \dfrac{(z_0^* + \Delta z^*) - z_0^*}{\Delta z} = \lim\limits_{\Delta z\to 0} \dfrac{\Delta z^*}{\Delta z}$, approaching z_0 parallel to the real axis, $\Delta z = \Delta x$, the result is 1. If, on the other hand, z_0 is approached parallel to the imaginary axis, $\Delta z = i\Delta y$ and the limit is -1. Thus the limit does not exist and the function is not differentiable at any point.

Example 2. For the function $f(z) = |z|^2$,

$$\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z_0+\Delta z)(z_0^* + \Delta z^*) - z_0^* z_0}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \left(z_0^* + \Delta z^* + z_0 \frac{\Delta z^*}{\Delta z} \right)$$

Again, if we approach z_0 parallel to the real axis the limit is $(z_0^* + z_0)$, while approaching z_0 parallel to the imaginary axis one gets $(z_0^* - z_0)$. Therefore, the limit does not exist and the function is not differentiable at z_0. An exception, however, occurs in the case $z_0 = 0$. Then the above limit reduces to $\lim\limits_{\Delta z\to 0} \Delta z^* = 0$. The function $f(z) = |z|^2$ is differentiable only at the point $z_0 = 0$.

Definition

A function is said to be *analytic* at a point z_0, if it is differentiable at z_0 and is also differentiable at every point in some neighbourhood of z_0.

The condition for analyticity is therefore very severe. $f(z) = z^2$, e^z, $\sin z$ are some examples of analytic functions. The function $f(z) = 1/z$ is analytic everywhere except at $z = 0$. On the other hand, the function $f(z) = |z|^2$ is not analytic at any point—not even at the point $z_0 = 0$, although it is differentiable there. If a function is analytic everywhere in the entire z-plane, it is called an *entire function*. An analytic function is sometimes referred to as a *holomorphic function*.

1.3 Cauchy-Riemann conditions

If the derivative of a function exists at a point, the real and imaginary parts of the function must satisfy certain conditions. Let $f(z) = u(x, y) + iv(x, y)$. If the derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ exists, then

$$\left.\frac{\partial u}{\partial x}\right|_{x_0,y_0}=\left.\frac{\partial v}{\partial y}\right|_{x_0,y_0} \tag{1.10a}$$

$$\left.\frac{\partial u}{\partial y}\right|_{x_0,y_0}=-\left.\frac{\partial v}{\partial x}\right|_{x_0,y_0} \tag{1.10b}$$

Equations (1.10a) and (1.10b) are known as *Cauchy-Riemann conditions.*

***Proof*:** Let us calculate the derivative approaching z_0 parallel to the real axis. Then $\Delta z = \Delta x$ and

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{\Delta x\to 0}\frac{[u(x_0+\Delta x, y_0)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0)-v(x_0,y_0)]}{\Delta x}$$

$$= \left.\frac{\partial u}{\partial x}\right|_{x_0,y_0}+i\left.\frac{\partial v}{\partial x}\right|_{x_0,y_0} \tag{1.11}$$

If, on the other hand, z_0 is approached parallel to the imaginary axis, $\Delta z = i\Delta y$, and

$$f'(z_0) = \lim_{\Delta y\to 0}\frac{[u(x_0, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0, y_0+\Delta y)-v(x_0,y_0)]}{i\Delta y}$$

$$= -i\left.\frac{\partial u}{\partial y}\right|_{x_0,y_0}+\left.\frac{\partial v}{\partial y}\right|_{x_0,y_0} \tag{1.12}$$

Since the derivative exists, the right hand sides of (1.11) and (1.12) must be equal.

$$\left.\frac{\partial u}{\partial x}\right|_{x_0,y_0}+i\left.\frac{\partial v}{\partial x}\right|_{x_0,y_0}=-i\left.\frac{\partial u}{\partial y}\right|_{x_0,y_0}+\left.\frac{\partial v}{\partial y}\right|_{x_0,y_0} \tag{1.13}$$

Equating the real and imaginary parts from both sides of (1.13) one gets the set of equations given in (1.10).

For convenience of writing, from now on, we shall omit the subscripts x_0, y_0 in the derivatives $\left.\frac{\partial u}{\partial x}\right|_{x_0,y_0}$, $\left.\frac{\partial v}{\partial y}\right|_{x_0,y_0}$ etc. Differentiating (1.10a) w.r.t. x, keeping y fixed one gets;

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \tag{1.14}$$

Similarly, from (1.10b) we obtain

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x} \tag{1.15}$$

In (1.14) and (1.15), it has been assumed that the second partial derivatives exist. If, in addition, the derivatives $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous, $\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}$ and hence

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1.16}$$

Similarly, differentiating (1.10a) w.r.t y holding x fixed and (1.10b) w.r.t x keeping y fixed, one obtains

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{1.17}$$

Equations (1.16) and (1.17) are Laplace's equation (Chapter 2) in two dimensions. A real function of x and y having continuous first and second partial derivatives and satisfying two dimensional Laplace's equation is known as a harmonic function. The real and imaginary parts of an analytic function are harmonic.

The Cauchy-Riemann conditions help us determine whether a function is differentiable at a point or not. Let us consider the functions discussed in Examples 1 and 2. For

$$f(z) = z^* = x - iy,\ u = x,\ v = -y.$$

Thus, $$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y},$$

showing that the Cauchy-Riemann conditions are not satisfied at any point. For the function $f(z) = |z|^2 = x^2 + y^2$, $u = x^2 + y^2$ and $v = 0$. Thus $\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = 0, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial x} = 0$ and Cauchy-Riemann conditions can be satisfied at the point $x = 0, y = 0$, but cannot be satisfied at any other point.

If the Cauchy-Riemann conditions are not satisfied at a point, the derivative does not exist and the function is not analytic at that point. Thus $f(z) = z^*$ is not analytic anywhere in the z-plane. But the satisfaction of the Cauchy-Riemann conditions does not necessarily imply the existence of the derivative as is illustrated in the following example.

Example 3. Consider the function,

$$f(z) = \frac{(z^*)^2}{z} \quad \text{for} \quad z \neq 0$$
$$= 0 \quad \text{for} \quad z = 0$$

For $\quad z \neq 0$

$$f(z) = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{y^3 - 3x^2y}{x^2 + y^2}$$

Therefore,

$$\left.\begin{aligned} u(x, y) &= \frac{x^3 - 3xy^2}{x^2 + y^2}, \quad && x \neq 0, y \neq 0 \\ &= 0, && x = y = 0 \end{aligned}\right\} \tag{1.18}$$

and

$$\left.\begin{aligned} v(x, y) &= \frac{y^3 - 3x^2y}{x^2 + y^2}, \quad && x \neq 0, y \neq 0 \\ &= 0, && x = y = 0 \end{aligned}\right\} \tag{1.19}$$

From (1.18) and (1.19), it is easy to obtain $\frac{\partial u}{\partial x}(0, 0) = 1$, $\frac{\partial u}{\partial y}(0, 0) = 0$, $\frac{\partial v}{\partial x}(0, 0) = 0$ and $\frac{\partial v}{\partial y}(0, 0) = 1$ so that the Cauchy-Riemann conditions are satisfied at $z = 0$.

But,
$$\lim_{\Delta z \to 0} \frac{\frac{(\Delta z^*)^2}{\Delta z} - 0}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta z^*}{\Delta z}\right)^2$$

If the point $z = 0$ is approached along the line $y = mx$

$$\Delta z = \Delta x + i\Delta y = (1 + im)\Delta x, \text{ and}$$

$$\lim_{\Delta z \to 0} \left(\frac{\Delta z^*}{\Delta z}\right)^2 = \lim_{\Delta x \to 0} \frac{(1 - im)^2(\Delta x)^2}{(1 + im)^2(\Delta x)^2} = \frac{(1 - im)^2}{(1 + im)^2}$$

The limit obviously depends on the line along which the point $z = 0$ is approached and therefore does not exist. The function is not differentiable at $z = 0$.

It can be shown that if the Cauchy-Riemann conditions are satisfied and in addition, the partial derivatives of u and v are continuous at a point then the function is differentiable at that point. One can verify that these conditions are not satisfied in Example 3.

For the function $f(z) = |z|^2$, the Cauchy-Riemann conditions are satisfied at $x = 0, y = 0$ and also, the partial derivatives of u and v are continuous at the point. Therefore, the function is differentiable at $x = 0, y = 0$. But this being the only point at which it is differentiable, the function is not analytic at $z = 0$.

1.4 Cauchy integral theorem

Integrals involving complex functions can be introduced as follows. A curve $y = y(x)$ in the complex plane will be called piecewise smooth, if (1) $y(x)$ is continuous, and (2) $\frac{dy}{dx}$ is continuous except at a certain finite set of points where it changes discontinuously. Let C be a piecewise smooth curve in the z-plane. Then

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \tag{1.20}$$

dz in (1.20) has to be along curve C. The direction in which the curve is traversed has to be specified. If the curve C is closed it is called a contour and the integral is denoted by $\oint_C f(z)\,dz$.

The integral $\int_C f(z)\,dz$ thus can be expressed as a sum of line integrals involving real variables. Green's theorem in connection with this will be useful later.

Green's Theorem

If the real function $P(x, y)$ and $Q(x, y)$ and their first partial derivatives are continuous inside a simply connected region and C is a piecewise smooth simple curve inside the region, then

$$\int_C (P\,dx + Q\,dy) = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) ds \tag{1.21}$$

where, S is the area enclosed by C.

A curve is said to be simple if it does not intersect itself. A simple closed curve is a simple curve joined end to end.

A simply connected domain is a domain such that every simple closed curve in it encloses only points inside the domain. The annular region between two circles in Fig. 1.2 is not simply connected because the simple closed curve C encloses the shaded region which is outside the domain.

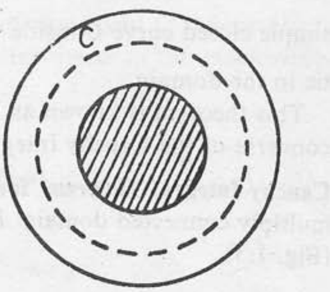

Fig. 1.2

We now state a very important theorem involving integrals of analytic functions.

Theorem

If $f(z)$ is analytic at all points within a simply connected region, and C is a piecewise smooth simple curve in it, then

$$\oint_C f(z)\,dz = 0 \tag{1.22}$$

This theorem is known as the *Cauchy integral theorem.*

Poof: The line integral $\oint_C f(z)\,dz$ can be written as

as $$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \tag{1.20}$$

Since $f(z)$ is analytic, its real and imaginary parts u and v and their first partial derivatives are continuous inside the simply connected region. Therefore, Green's theorem can be applied to both the integrals on the right hand side of (1.20)

$$\oint_C (u\,dx - v\,dy) = \int_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) ds \tag{1.23}$$

and $$\oint_C (v\,dx + u\,dy) = \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) ds \tag{1.24}$$

Since $f(z)$ is analytic, u and v satisfy Cauchy-Riemann conditions (1.10a) and (1.10b) by virtue of which the right hand sides of both (1.23) and (1.24) are zero. Hence $\oint_C f(z)\,dz = 0$.

Analyticity of $f(z)$ leads to the result $\oint_C f(z)\,dz = 0$. On the other hand, if it is given that the integral is zero, can we conclude that the function is analytic under certain conditions ? Yes, we can.

Theorem

If a function is continuous in a simply connected domain and if, for each simple closed curve C inside the domain $\oint_C f(z)\,dz = 0$, then, $f(z)$ is analytic in the domain.

This theorem is known as Morera's Theorem. In a certain sense, it is the converse of the Cauchy Integral Theorem.

Cauchy Integral Theorem for Multiply Connected Regions: Consider the multiply connected domain D enclosed between closed curves C_1 and C' (Fig. 1.3).

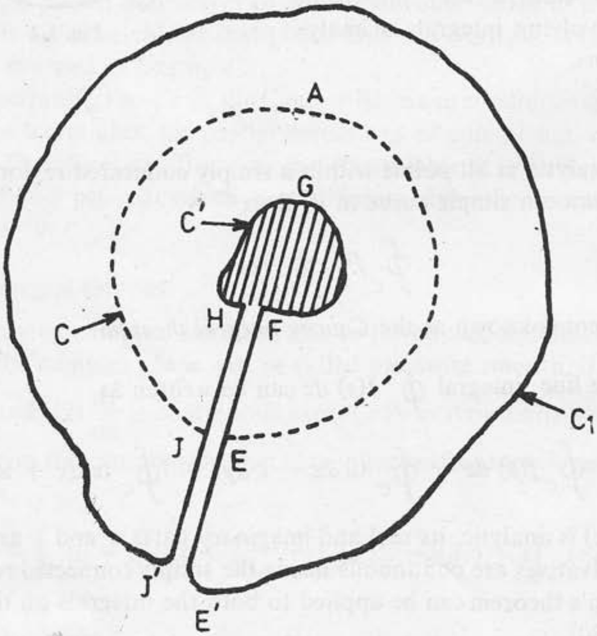

Fig. 1.3

Cauchy integral theorem cannot be applied to the function $f(z)$ which is analytic in D if the contour of integration encloses C'. Let us make a narrow cut by joining points H and F on C' to points J' and E' respectively on C_1 by straight lines. The gap between the lines is supposed to be narrow and will vanish in the limit. With the cut, if the set of points between the lines is excluded, the domain D becomes simply connected. Consider the closed curve $C \equiv AJHGFEA$ in the domain. By Cauchy integral theorem,

$$\oint_C f(z)\,dz = 0$$

or, $$\int_{EAJ} f(z)\,dz + \int_{JH} f(z)\,dz + \int_{HGF} f(z)\,dz + \int_{FE} f(z)\,dz = 0 \qquad (1.25)$$

In the limit, if the width of the strip between the straight lines is made zero, then

$$\int_{JH} f(z)\,dz + \int_{FE} f(z)\,dz \to 0$$

Also the curves EAJ and HGF become closed contours C and C', where C is traversed in the anticlockwise direction but C' is traversed in the clockwise direction. Thus (1.25) reduces to,

$$\oint_C f(z)\,dz + \oint_{C'} f(z)\,dz = 0 \qquad (1.26)$$

Reversing the direction of the second contour C', we get

$$\oint_C f(z)\,dz - \oint_{C'} f(z)\,dz = 0 \qquad (1.27)$$

This is the modified Cauchy integral theorem for a multiply connected region. Obviously, if the multiply connected domain is the region enclosed by closed curves C_1 and $C_1', C_2', \ldots, C_n'$ (Fig. 1.4), and if the contour of integration is inside C_1 and encloses all the other curves, the contributions corresponding to each of the closed curves $C_1'. C_2', , \ldots, C_n'$ inside C have to be taken into account. Then

$$\oint_C f(z)\,dz - \oint_{C_1'} f(z)\,dz - \oint_{C_2'} f(z)\,dz \ldots \oint_{C_n'} f(z)\,dz = 0 \qquad (1.28)$$

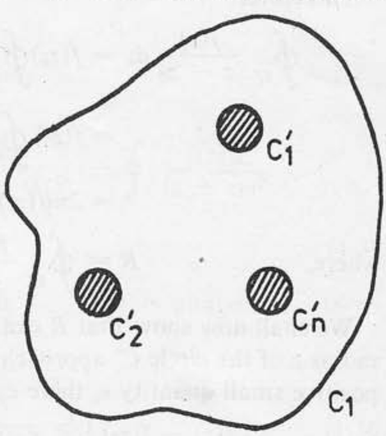

Fig. 1.4

1.5 Cauchy Integral Formula

The positive sense of a closed contour is defined as the direction such that the area enclosed always remains to the left of the contour.

Theorem

If $f(z)$ is analytic in a domain D, C a simple closed contour in the domain, and z_0 a point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz \tag{1.29}$$

where the contour C is taken in the positive sense.

This theorem goes by the name Cauchy integral formula.

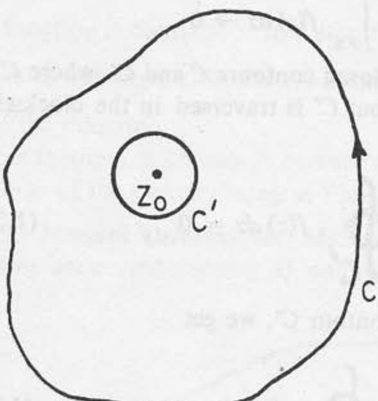

Fig. 1.5

Proof. The function $\dfrac{f(z)}{z - z_0}$ is analytic everywhere on C and inside C except at the point z_0. Let us draw a circle C' of radius r around z_0 such that it lies inside C. The region between C and C' is multiply connected and the function $\dfrac{f(z)}{z - z_0}$ is analytic everywhere inside it and also on C. Applying the modified Cauchy integral theorem to the function, we get

$$\oint_C \frac{f(z)}{z - z_0}dz - \oint_{C'} \frac{f(z)}{z - z_0}\,dz = 0 \tag{1.30}$$

where both the contours are in the positive sense. But on C' $z - z_0 = re^{i\theta}$ and $dz = ire^{i\theta}\,d\theta$.

Therefore,

$$\oint_{C'} \frac{f(z)}{z - z_0}\,dz = f(z_0)\oint_{C'} \frac{dz}{z - z_0} + \oint_{C'} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

$$= f(z_0)\oint_{C'} \frac{ire^{i\theta}d\theta}{re^{i\theta}} + \oint_{C'} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

$$= 2\pi i f(z_0) + R \tag{1.31}$$

where,
$$R = \oint_{C'} \frac{f(z) - f(z_0)}{z - z_0}\,dz \tag{1.32}$$

We shall now show that R can be made arbitrarily small by making the radius r of the circle C' approach zero. From the continuity of $f(z)$, for every positive small quantity ϵ, there exists a δ such that

$$|f(z) - f(z_0)| < \epsilon, \text{ whenever } |z - z_0| < \delta \tag{1.33}$$

From (1.30) and (1.31), we have

$$R = \oint_C \frac{f(z)}{z - z_0}\,dz - 2\pi i f(z_0) \tag{1.34}$$

Since the right hand side does not depend on r, R is also independent of the radius of the circle around z_0. Let us choose r such that $|z - z_0| < \delta$ on the circle. From the continuity of $f(z)$ at z_0, $|f(z) - f(z_0)| < \epsilon$, whenever z is on the circle C'. Therefore, from (1.32)[1]

$$|R| \leqslant \oint_{C'} \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz|$$

$$\leqslant \epsilon \oint_{C'} \frac{|dz|}{|z - z_0|} = 2\pi\epsilon \tag{1.35}$$

and can be made arbitrarily small. Therefore, $|R| = 0$

A direct consequence of Cauchy integral formula (1.29) is that for an analytic function, the derivative of $f(z)$ at z_0 exists and is given by

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz \tag{1.36}$$

Since, $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ from (1.29), we obtain

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{1}{2\pi i} \cdot \frac{1}{\Delta z} \left[\oint_C \frac{f(z)\, dz)}{z - z_0 - \Delta z} - \oint_C \frac{f(z)\, dz}{z - z_0} \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)(z - z_0)} \tag{1.37}$$

From the continuity of $f(z)$ on C, one can show that the above limit $= \dfrac{1}{2\pi i} \oint_C \dfrac{f(z)\, dz}{(z - z_0)^2}$ thus leading to (1.36). The derivative $f'(z_0)$ exists because the integrand $\dfrac{f(z)}{(z - z_0)^2}$ in (1.36) exists at every point on C. The result can be easily generalized.

Thus,

$$f''(z_0) = \lim_{\Delta z \to 0} \frac{f'(z_0 + \Delta z) - f'(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \cdot \frac{1}{\Delta z} \left[\oint_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)^2} - \oint_C \frac{f(z)\, dz}{(z - z_0)^2} \right]$$

$$= \frac{1}{2\pi i} \cdot 2 \cdot \oint_C \frac{f(z)}{(z - z_0)^3} dz \tag{1.38}$$

Since, $f''(z_0)$ exists at every point z_0 inside C, $f'(z_0)$ is analytic inside the contour.

The n^{th} derivative is given by

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \cdot n! \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{1.39}$$

Thus we arrive at the following remarkable theorem:

---

[1] We use the result $\left| \int_C (fz)\, d(z) \right| \leqslant \int_C |f(z)|\, |dz|$, which is a consequence of the triangular inequality (1.5).

Theorem

If a function $f(z)$ is analytic at the point z_0, its derivatives of all orders exist at the point and are given by

$$f^{(n)}(z_0) = \frac{1}{2\pi i}\cdot n! \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

The derivatives are also analytic at the point.

Another celebrated theorem which is a consequence of Cauchy integral formula is Liouville's theorem which is as follows:

Theorem

If a function is analytic everywhere in the complex plane and is bounded, it is a constant function.

Proof: Consider a circle C of radius r around an arbitrary point z_0. From (1.36), one gets

$$|f'(z_0)| \leqslant \frac{1}{2\pi}\oint_C \frac{|f(z)|}{|z - z_0|^2}\,|dz| \tag{1.40}$$

Since $f(z)$ is bounded $|f(z)| < M$, a given positive quantity. Then from (1.40), we obtain

$$\begin{aligned} |f'(z_0)| &\leqslant \frac{1}{2\pi}M\oint_c \frac{|dz|}{|z - z_0|^2} \\ &= \frac{1}{2\pi}M\frac{2\pi}{r} \end{aligned} \tag{1.41}$$

The function $f(z)$ being analytic everywhere in the complex plane, r can be made arbitrarily large without running into difficulty with the conclusion (1.41). Therefore $|f'(z_0)| = 0$. or $f'(z_0) = 0$. But z_0 is an arbitrary point. Therefore, the derivative of $f(z)$ is zero at all points of the complex plane and the function is a constant function.

1.6 Taylor and Laurent Series

Certain preliminary notions about the convergence of an infinite series of complex numbers will be needed before we can introduce the Taylor series of a function.

Definition

An infinite series of complex numbers of the form $z_1 + z_2 + \ldots + z_n + \ldots$ is said to *converge* to a sum S if the sequence, $S_N = \sum_{n=1}^{N} z_n$, $N = 1, 2 \ldots,$ converges to S.

Definition

The sequence S_N of complex numbers is said to converge to S if, for any given positive number ϵ there exists a positive integer N such that

$$| S - S_N | < \epsilon, \qquad \text{whenever} \qquad N > N_0.$$

In other words, for a converging sequence, the numbers S_N come arbitrarily close to S as N becomes very large, where $| S - S_N |$ is the measure of closeness. If a series does not converge, it is said to diverge.

An infinite series of the form,

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_n(z - z_0)^n + \ldots$$

where, $z_0, a_0, a_1, \ldots, a_n, \ldots$ are given complex numbers and z is a variable, is known as a power series.

Theorem

If $f(z)$ is analytic everywhere inside a circle C_0 centered at z_0, then for every point z inside C_0, the power series

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots$$

converges to $f(z)$.

The series can thus be represented as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots \quad (1.42)$$

for $| z - z_0 | < r$, where r is the radius of C_0.

Proof: Around z_0 we draw another circle C' of radius $r' < r$. Let z' denote any point on C' and let the point z be inside C'.

Since $f(z)$ is analytic on and inside C', by Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{z' - z}\, dz' \quad (1.43)$$

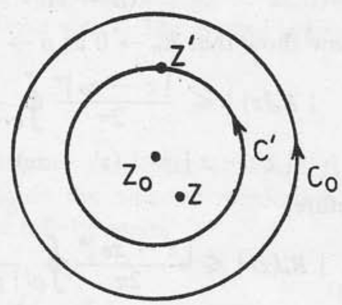

Fig. 1.6

Now

$$\frac{1}{z' - z} = \frac{1}{(z' - z_0) - (z - z_0)}$$

$$= \frac{1}{z' - z_0} \cdot \frac{1}{1 - z_1} \quad (1.44)$$

where,

$$z_1 = \frac{z - z_0}{z' - z_0}.$$

(1.43) can thus be rewritten in the form:

$$f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{z'-z_0}\cdot\frac{1}{1-z_1}\,dz' \tag{1.45}$$

From the equality,

$$1 + z_1 + z_1^2 + \ldots + z_1^{n-1} = \frac{1-z_1^n}{1-z_1}$$

we get,

$$\frac{1}{1-z_1} = 1 + z_1 + z_1^2 + \ldots + z_1^{n-1} + \frac{z_1^n}{1-z_1} \tag{1.46}$$

Substitution of (1.46) in (1.45) gives

$$f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{z'-z_0}\,dz' + \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{z'-z_0}\cdot\frac{(z-z_0)}{(z'-z_0)}\,dz' + \ldots$$

$$+ \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{z'-z_0}\cdot\frac{(z-z_0)^{n-1}}{(z'-z_0)^{n-1}} + R_n(z) \tag{1.47}$$

where,

$$R_n(z) = \frac{1}{2\pi i}\oint_{c'} \frac{f(z')}{z'-z_0}\cdot\frac{(z-z_0)^n}{(z'-z_0)^n}\cdot\frac{(z'-z_0)}{(z'-z)}\,dz'$$

$$= \frac{(z-z_0)^n}{2\pi i}\oint_{c'} \frac{f(z')}{(z'-z_0)^n(z'-z)}\,dz' \tag{1.48}$$

Using Eqn. (1.39) for the derivatives of $f(z)$ one gets from (1.47),

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \ldots$$

$$\ldots + \frac{(z-z_0)^{n-1}}{(n-1)!}f^{(n-1)}(z_0) + R_n(z) \tag{1.49}$$

We now show that $R_n \to 0$ as $n \to \infty$

$$|R_n(z)| \leqslant \frac{|z-z_0|^n}{2\pi}\oint_{C'} \frac{|f(z')|\cdot|dz'|}{|z'-z_0|^n\cdot|z'-z|} \tag{1.50}$$

But, $\quad |z'-z| = |(z'-z_0) + (z_0-z)| \geqslant \big||z'-z_0| - |z_0-z|\big|$.

Therefore,

$$|R_n(z)| \leqslant \frac{|z-z_0|^n}{2\pi}\oint_{C'} \frac{|f(z')|\cdot|dz'|}{|z'-z_0|^n\cdot\big||z'-z_0| - |z_0-z|\big|} \tag{1.51}$$

Writing, $z'-z_0 = r'e^{i\theta'}$, one obtains $|z'-z_0| = r'$ and $|dz'| = r'\,d\theta'$.

Thus,

$$|R_n(z)| \leqslant \frac{|z-z_0|^n}{2\pi}\cdot\oint_{C'} \frac{|f(z')|\,r'\,d\theta'}{(r')^{n+1}\left|1-\frac{|z-z_0|}{r'}\right|} \tag{1.52}$$

If M is the largest value of $|f(z')|$ on C', we have

$$| R_n(z) | \leqslant \frac{| z - z_0 |^n}{2\pi} \cdot \frac{M}{(r')^n\left(1 - \frac{| z_0 - z |}{r'}\right)} \cdot \oint_{C'} d\theta'$$

$$= \left(\frac{| z - z_0 |}{r'}\right)^n \cdot \frac{M}{1 - \frac{| z - z_0 |}{r'}} \tag{1.53}$$

Since, z is interior to C', $| z - z_0 | < r'$ and $R_n \to 0$ as $n \to \infty$.

Thus, if $f(z)$ is analytic at a point it can be expressed as a Taylor series around the point. The series is guaranteed to converge within a circle in which $f(z)$ is analytic. If $f(z)$ is expressed as a Taylor series around z_0 and z_1 is the nearest point from z_0 at which $f(z)$ is not analytic, the series is convergent at all points within a circle of radius $| z_0 - z_1 |$. This radius is called the radius of convergence.

Familiar examples of Taylor's series are

(1) $$e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z}{n!} + \ldots \tag{1.54}$$

This is a series around $z_0 = 0$. The series converges for $| z | < \infty$.

(2) $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \tag{1.55}$$

The series converges for $| z | < \infty$

(3) $$\frac{1}{1 - z} = 1 + z + z^2 + \ldots \tag{1.56}$$

The radius of convergence in this case is $| z | = 1$.

The Taylor series was developed for a function which is analytic at every point inside a circle C_0. If, however, the function fails to be analytic at certain points, can we develop it in an infinite series? The answer is given in the following theorem.

Theorem

Let C_0 and C_0' be two concentric circles around z_0. If the function $f(z)$ is analytic on C_0, C_0 and at every point inside the annular region between C_0 and C_0', then it can be expressed by the infinite series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{1.57}$$

where, $$a_n = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z')}{(z' - z_0)^{n+1}} dz', \quad n = 0, 1, 2, \ldots \tag{1.58}$$

$$b_n = \frac{1}{2\pi i} \oint_{C_0'} \frac{f(z')}{(z' - z_0)^{-n+1}} dz', \quad n = 1, 2, \ldots \tag{1.59}$$

Series (1.57) is valid at each point in the annular region and is known as the *Laurent series.*

Both the contours C_0 and C_0' are to be taken in the positive sense. Note that a_n in (1.58) cannot be equated to $\frac{f^{(n)}(z_0)}{n!}$, because it is not known whether $f(z)$ is analytic everywhere inside C_0. If $f(z)$ happens to be analytic everywhere inside C_0, $a_n = \frac{f^{(n)}(z_0)}{n!}$. Also then the function $f(z')/(z'-z_0)^{-n+1}$, $n = 1, 2, \ldots$, is analytic everywhere on and inside C_0' and

$$b_n = \frac{1}{2\pi i} \oint_{C_0'} \frac{f(z')}{(z'-z_0)^{-n+1}}\, dz' = 0.$$

Therefore, in this case the Laurent series reduces to the Taylor series.

Proof: Let z be any point inside the annular region between C_0 and C_0' (Fig. 1.7). Draw a circle C around z such that it lies entirely in the annular region. By Cauchy integral theorem applied to the multiply connected region, we have,

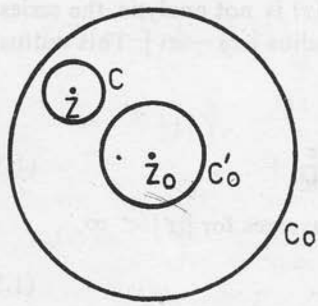

Fig. 1.7

$$\oint_{C_0} \frac{f(z')}{z'-z}\, dz' - \oint_{C_0'} \frac{f(z')}{z'-z}\, dz' - \oint_C \frac{f(z')}{z'-z}\, dz' = 0 \qquad (1.60)$$

where, all the contours are taken in the positive sense. But by Cauchy integral formula

$$\oint_C \frac{f(z')}{z'-z}\, dz' = 2\pi i f(z) \qquad (1.61)$$

Therefore, $$f(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z')}{z'-z}\, dz' - \frac{1}{2\pi i} \oint_{C_0'} \frac{f(z')}{z'-z}\, dz' \qquad (1.62)$$

In the first integral on the r.h.s., writing

$$\frac{1}{z'-z} = \frac{1}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{z'-z_0} + \frac{(z-z_0)}{(z'-z_0)^2} + \ldots + \frac{(z-z_0)^{N-1}}{(z'-z_0)^N} + \frac{(z-z_0)^N}{(z'-z_0)^{N+1}\cdot\left(1-\frac{z-z_0}{z'-z_0}\right)} \qquad \text{[cf. Eqn, (1.46)]}$$

we get,

$$\frac{1}{2\pi i} \oint_{C_0} \frac{f(z')}{(z'-z_0)}\, dz' = \sum_{n=0}^{N-1} a_n(z-z_0)^n + R_N(z) \qquad (1.63)$$

where, $$a_n = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

and, $$R_N = \frac{(z-z_0)^N}{2\pi i}\cdot\oint_{C_0}\frac{f(z')\,dz'}{(z'-z_0)^N\cdot(z'-z)} \tag{1.64}$$

As in the proof of Taylor series, we can show that $R_N \to 0$ as $N \to \infty$.

In the second integral on the r.h.s. of (1.62), writing

$$-\frac{1}{z'-z} = \frac{1}{(z-z_0)-(z'-z_0)}$$

$$= \frac{1}{z-z_0} + \frac{(z'-z_0)}{(z-z_0)^2} + \ldots + \frac{(z'-z_0)^{N-1}}{(z-z_0)^N}$$

$$+ \frac{(z'-z_0)^N}{(z-z_0)^{N+1}\left(1-\dfrac{z'-z^0}{z-z_0}\right)}$$

one obtains,

$$-\frac{1}{2\pi i}\oint_{C_0'}\frac{f(z')\,dz'}{z'-z} = \sum_{n=1}^{N}\frac{b_n}{(z-z_0)^n} + Q_N(z) \tag{1.65}$$

where, $$b_n = \frac{1}{2\pi i}\oint_{C_0'}\frac{f(z')}{(z'-z_0)^{-n+1}}\,dz'$$

and, $$Q_N(z) = \frac{1}{2\pi i}\oint_{C_0'}\frac{f(z')\cdot(z'-z_0)^N}{(z-z_0)^N\cdot(z-z')}\,dz' \tag{1.66}$$

Following an argument similar to that for $R_N(z)$ in case of Taylor's series, we can show that $Q_N(z) \to 0$ as $N \to \infty$. Thus

$$|\,Q_N(z)\,| \leqslant \frac{1}{2\pi\cdot|\,z-z_0\,|^N}\oint_{C_0'}\frac{|\,f(z')\,|\cdot|\,z'-z_0\,|^N}{|\,z-z'\,|}\,|\,dz'\,| \tag{1.67}$$

$$|\,z-z'\,| = |\,(z-z_0)+(z_0-z')\,| \geqslant |\,|\,z-z_0\,| - |\,z_0-z'\,|\,|.$$

Therefore

$$|\,Q_N(z)\,| \leqslant \frac{1}{2\pi\cdot|\,z-z_0\,|^N}\oint_{C_0'}\frac{|\,f(z')\,|\cdot|\,z'-z_0\,|^N}{|\,|\,z-z_0\,|-|\,z'-z_0\,|\,|}\,|\,dz'\,| \tag{1.68}$$

Writing, $z'-z_0 = r'e^{i\theta'}$ on C_0', we obtain

$$|\,Q_N(z)\,| \leqslant \frac{1}{2\pi\cdot|\,z-z_0\,|^{N+1}}\oint_{C_0'}\frac{|\,f(z')\,|\,(r')^N}{\left(1-\dfrac{r'}{|\,z-z_0\,|}\right)}\,r'\,d\theta' \tag{1.69}$$

If M is the maximum value of $|\,f(z')\,|$ on C_0', it follows from (1.69) that

$$|\,Q_N(z)\,| \leqslant M\cdot\left(\frac{r'}{|\,z-z_0\,|}\right)^{N+1}\cdot\frac{1}{1-\dfrac{r'}{|\,z-z_0\,|}} \tag{1.70}$$

Since the point z is exterior to C_0', $|\,z-z_0\,| > r'$, the radius of C_0'. Therefore, $|\,Q_N(z)\,| \to 0$ as $N \to \infty$, and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

where the coefficients a_n and b_n are given by (1.58) and (1.59) respectively.

Laurent series can be obtained without evaluating the integrals (1.58) and (1.59), by using Taylor series for part of the function $f(z)$.

Example 4. As an illustration, consider the Laurent series of the function $f(z) = 1/z^2(1 - z)$ which is analytic everywhere except at $z = 1$ and $z = 0$. For $|z| < 1$, the function $\frac{1}{1-z}$ can be expanded in a Taylor series:

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots \tag{1.71}$$

Therefore,
$$f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + \ldots \tag{1.71}$$

The result (1.71) is a Laurent series for the given function with $b_1 = 1$, $b_2 = 1$, $b_3 = b_4 \ldots = 0$, $a_0 = a_1 = \ldots = a_n = \ldots = 1$. The Laurent series representation (1.71) is valid for $|z| < 1, z \neq 0$.

If we want the Laurent series for the same function for $|z| > 1$, $f(z)$ has to be rewritten as

$$f(z) = -\frac{1}{z^3\left(1 - \frac{1}{z}\right)}$$

For $|z| > 1$, $\left|\frac{1}{z}\right| < 1$, and the expression $\frac{1}{1-\frac{1}{z}}$ can be expanded in a Taylor series, as

$$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots$$

which converges for $|z| > 1$. Therefore, the desired Laurent series for the given function is given by

$$f(z) = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} \ldots \tag{1.72}$$

Thus, $b_1 = b_2 = 0$, $b_3 = b_4 = \ldots = -1$, $a_1 = a_2 = \ldots = 0$.

The series representation (1.71) is valid inside a circle of radius unity about the origin (except at $z = 0$). On the other hand, (1.72) is valid outside the circle. It can be said that (1.72) is the *analytic continuation* of (1.71) outside the unit circle around the origin.

Example 5. Let us find the Laurent series representation of the function $f(z) = \frac{1}{e^z - 1}$. $z = 0$, and $z = \pm 2\pi i n$, $n = 1, 2, \ldots$ are points at which $f(z)$ fails to be analytic.

$$f(z) = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots} = \frac{1}{z} \cdot \frac{1}{1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \ldots} \tag{1.73}$$

The denominator of $\dfrac{1}{1+\dfrac{z}{2}+\dfrac{z^2}{3!}+\cdots}$ is not zero for $|z|<2\pi$. Therefore, the Taylor series for it about $z=0$ will converge within the circle $|z|<2\pi$.

$$\frac{1}{1+\dfrac{z}{2!}+\dfrac{z^2}{3!}+\dfrac{z^3}{4!}+\cdots}=1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}\right)$$

$$+\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}\right)^2-\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}\right)^3+\cdots$$

$$=1-\frac{z}{2}+\left(-\frac{1}{3!}+\frac{1}{4}\right)z^2+\left(-\frac{1}{4!}+\frac{1}{3!}-\frac{1}{8}\right)z^3$$

$$+\left(-\frac{1}{5!}+\frac{1}{(3!)^2}+\frac{1}{4!}-\frac{1}{8}+\frac{1}{16}\right)z^4+\cdots$$

$$=1-\frac{z}{2}+\frac{1}{12}z^2-\frac{1}{720}z^4+\cdots$$

Thus $$f(z)=\frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\cdots \tag{1.74}$$

The representation is valid for $0<|z|<2\pi$.

1.7 Singularities and their Classification

Definition

If $f(z)$ is not analytic at z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is called a singularity of the function $f(z)$.

In Example 4, the function $f(z)=\dfrac{1}{z^2(1-z)}$ is analytic at every point except at $z=0$ and $z=1$. Therefore, they are the singularities of $f(z)$.

As another example, consider the function $f(z)=\dfrac{1}{\sinh z}$. This function is analytic everywhere except at the points where $\sinh z=0$. This happens when $e^z-e^{-z}=0$ or, $e^{2z}=1=e^{i2\pi n}$, $n=0,\pm1,\pm2,\ldots$. Thus the singularities of the function are $z=0,\pm i\pi,\pm 2i\pi,\ldots$.

A singularity z_0 will be called *Isolated Singularity* if we can find a neighbourhood of z_0 in which it is singular only at z_0 but is analytic at every other point. All the singularities considered above are examples of isolated singularities. In contrast consider the singularities of $f(z)=\dfrac{1}{\sin\left(\dfrac{1}{z}\right)}$, $z=\dfrac{1}{n\pi}$, $n=\pm1,\pm2,\ldots$, and $z=0$ are the singular points. Every neighbourhood of $z = 0$ will include other singularities of $f(z)$ corresponding to $z = \frac{1}{n\pi}$ with a very large value of $|n|$. Thus, the singularity at $z = 0$ is not isolated, but all the other singularities are.

Definition

If a function $f(z)$ is not analytic at z_0 but there exists a positive integer m such that $\lim_{z \to z_0} (z - z_0)^m f(z) \equiv \phi(z_0)$ exists and $\phi(z_0) \neq 0$, then $z = z_0$ is said to be a *Pole of Order m* of $f(z)$.

A pole of order one is called a *simple pole*. In Example 4, $z = 0$ is a pole of order two of the function $f(z) = \frac{1}{z^2(1 - z)}$ and $z = 1$ is a simple pole. For the function $f(z) = \frac{1}{\sinh z}$, $\lim_{z \to in\pi} (z - in\pi) \frac{1}{\sinh z} = (-1)^n$, where n is an integer. Therefore $z = in\pi$ is a simple pole of the function.

If z_0 is an isolated singularity of $f(z)$, we can represent $f(z)$ by a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{1.75}$$

which is valid in some neighbourhood around $z_0 : 0 < |z - z_0| < r$. The part of the series containing only negative powers of $(z - z_0)$ is called the *Principal Part*. It is clear that if the summation over n in the principal part runs upto ∞, there will exist no positive integer m such that $\lim_{z \to z_0} (z - z_0)^m f(z)$ will be finite. In such a case z_0 is called an *Essential Singular Point* of $f(z)$. On the other hand, the limit will be finite and not equal to zero, if the principal part of the function is of the form,

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_m}{(z - z_0)^m}.$$

Thus, we can give the following alternative definition of a pole of order m:

If the principal part of $f(z)$ around z_0 contains a finite number of terms and m is the highest negative power of $(z - z_0)$, then z_0 is called a pole of order m of $f(z)$.

It is sometimes possible that a function fails to be analytic at $z = z_0$, but the principal part of the function at z_0 is zero, i.e., $b_1 = b_2 = \ldots = 0$. $f(z) = \frac{\sin z}{z}$ is an example of such a function. Using the Taylor series of $\sin z$, we get $f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots$ The expansion is valid for $|z| < \infty$ except at $z = 0$ which is a singular point. Such a singularity is known as *removable singularity*. The function can be made analytic at $z=0$ by defining $f(0) = 1$.

Definition

The coefficient of $\frac{1}{(z - z_0)}$ in the principal part of the Laurent series expansion of a function $f(z)$ around and isolated singular point z_0 is called the *Residue* of the function at z_0.

In (1.75), b_1 is the residue of $f(z)$ at z_0. From (1.59), b_1 is given by

$$b_1 = \frac{1}{2\pi i} \oint_{C_0'} f(z)\, dz \tag{1.76}$$

This gives us a method of evaluating the integral $\oint_{C_0'} f(z)\, dz$ if the residue b_1 is known. Before proceeding further to the residue theorem and the evaluation of integrals using it, we discuss a different type of singularity associated with multi-valued functions.

Branch cut and branch point

Consider again the function $f(z) = \log_e z = \log_e r + i\theta,\ -\pi < \theta \leqslant \pi$. The two points $z_1 = re^{i(\pi - \epsilon)}$ and $z_2 = re^{-i\pi + i\epsilon}$ lie very close to the negative x-axis (Fig. 1.8) and as $\epsilon \to 0$ they approach the same point. But $\lim_{\epsilon \to 0} f(z_1) = \log_e r + i\pi$ and $\lim_{\epsilon \to 0} f(z_2) = \log_e r - i\pi$. As we approach the negative real axis from above and from below, the function approaches different limits. The function is not continuous on the negative x-axis and is, therefore, not differentiable at any of these points. Also, $f(z)$ is not defined for $z = 0$. Thus all the points on the negative real axis including $z = 0$ are singular points of the function. The line $\theta = -\pi$ on which the function fails to be analytic is called a *branch cut*. The function $\log_e z = \log_e r + i\theta, -\pi < \theta < \pi$ is analytic in the given domain and is known as the *principal branch* of $\log_e z$.

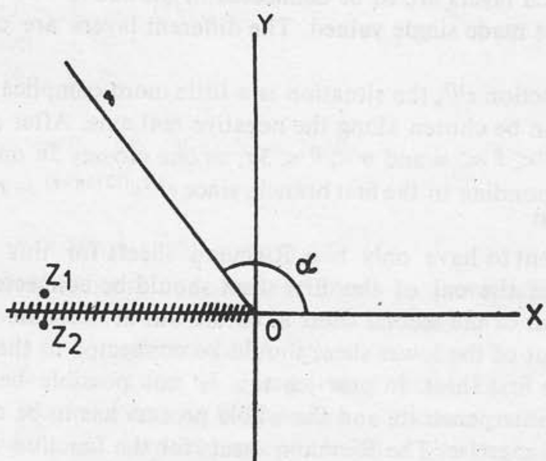

Fig. 1.8

If we define the function in a different way, i.e., $\log_e z = \log_e r + i\theta$, $-(2\pi - \alpha) < \theta \leqslant \alpha$, the function will be discontinuous at all points on the line $\theta = \alpha$. This line will then be the branch cut. For a multiple-valued function, there is thus a branch cut which is in general, a curve of singular points. The position of the cut in the z-plane is somewhat arbitrary depending on the definition of the branch of the multiple-valued function. However, there will be a point which is common to all the cuts. For the function. $\log_e z$, $z = 0$ is such a point. This particular singular point is called a *Branch Point*. Incidentally, this is not an isolated singularity. For $\log_e z = \log_e r + i\theta$, $-\pi < \theta \leqslant \pi$, every neighbourhood of $z = 0$ will contain points on the negative x-axis which are also singular points of the function.

The function $f(z) = z^{1/2} = r^{1/2}e^{i\theta/2}$ is another example of branch point singularity. If the domain of definition of θ is $-\pi < \theta \leqslant \pi$, then the negative real axis is the branch cut. The cut can, however, be chosen along the positive real axis by defining the function for $0 \leqslant \theta < 2\pi$, $z = 0$ is the branch point.

To avoid the ambiguities associated with the definition of multiple valued functions, Riemann thought of an ingenious device. The technique is best illustrated with the help of examples. Consider the principal branch of the function $\log_e z = \log_e r + i\theta$, $-\pi < \theta < \pi$. Other branches are given by $\pi < \theta < 3\pi$, $3\pi < \theta < 5\pi$, etc. If we consider only a particular branch, the function is analytic. Following Riemann, we now consider each branch of the function to constitute a layer of the z - plane. The layers are stacked one above the other such that the top layer is the principal branch and the subsequent layers are in the order of increasing θ. Each of the branches has a cut along the negative real axis. The upper edge of the cut in the first layer is connected to the lower edge of the cut in the second layer. The upper edge of the cut in the second layer is connected to the lower edge of the cut in the third layer, and so on. For the function $\log_e z$, an infinite number of such layers are to be connected in the above way. By this device the function is made single valued. The different layers are called *Riemann sheets*.

For the function $z^{1/2}$, the situation is a little more complicated. Again, the branch cut can be chosen along the negative real axis. After considering the branches $-\pi < \theta < \pi$ and $\pi < \theta < 3\pi$, as one crosses 3π one gets back the values corresponding to the first branch, since $r^{1/2}e^{i/2\,(3\pi+\phi)} = r^{1/2}e^{i2\pi}e^{i/2(-\pi+\phi)} = r^{1/2}e^{i/2(-\pi+\phi)}$.

It is sufficient to have only two Riemann sheets for this function. The upper edge of the cut of the first sheet should be connected to the lower edge of the cut of the second sheet as before but at the same time the upper edge of the cut of the lower sheet should be connected to the lower edge of the cut of the first sheet. In practice this is not possible because the two connections interpenetrate and the whole process has to be considered as a mathematical exercise. The Riemann sheets for the function $z^{1/n}$, where n is an integer, can be constructed in a similar manner. For this function n Riemann sheets are required.

1.8 The Residue Theorem

Theorem

Let a simple closed curve C enclose a finite number of isolated singular points $z_1, z_2, \ldots, z_n$ of a function $f(z)$, which is analytic on C and at all other points interior to C. Then the integral $\oint_C f(z)\,dz$ taken in the positive sense, is given by

$$\oint_C f(z)\,dz = 2\pi i(R_1 + R_2 + \ldots + R_n) \tag{1.77}$$

where, R_i is the residue of $f(z)$ at the singularity z_i.

Proof: Let us draw circles $C_1, C_2, \ldots, C_n$ around the singular points $Z_1, Z_2, \ldots, Z_n$ respectively. The radii of the circles are so small that, (1) they lie entirely within the curve C, and (2) they do not overlap. Cauchy integral theorem applied to the multiply connected region gives

$$\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \ldots - \oint_{C_n} f(z)\,dz = 0 \tag{1.78}$$

where all the integrals are taken in the positive sense.

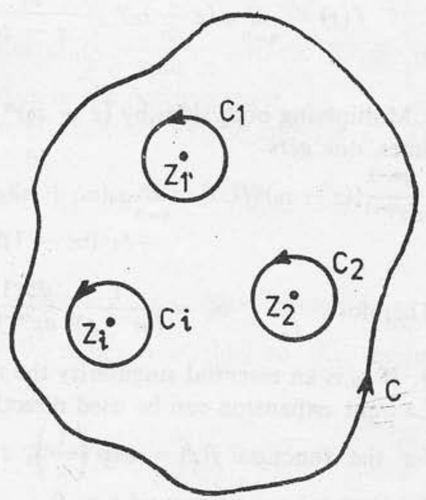

Fig. 1.9

But from (1.76),

$$\oint_{C_1} f(z)\,dz = 2\pi i R_1, \quad \oint_{C_2} f(z)\,dz = 2\pi i R_2, \ldots \oint_{C_n} f(z)\,dz = 2\pi i R_n \tag{1.79}$$

Substitution from (1.79) in (1.78) gives (1.77).

The residue theorem gives us a powerful method of evaluating integrals. To get the integral around a closed contour all we have to do is to note the singularities of the integrand enclosed by the contour and find the residues at the singularities. However, we cannot use Eqn. (1.76) to calculate the residues because this will bring us back to the problem of evaluation of the integral. Fortunately, the residues can be calculated by other methods.

CALCULATION OF RESIDUES

1. For a Simple Pole

If z_0 is a simple pole of $f(z)$, the Laurent expansion of $f(z)$ around z_0 valid for $0 < |z - z_0| < r$ is of the form,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} \tag{1.80}$$

Then, $\lim_{z \to z_0} (z - z_0) f(z) = b_1 =$ the residue of $f(z)$ at z_0.

2. For a Pole of Order m

If z_0 is a pole of order m of $f(z)$, the Laurent expansion of $f(z)$ around z_0 valid for $0 < |z - z_0| < r$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_m}{(z - z_0)^m} \tag{1.81}$$

Multiplying both sides by $(z - z_0)^m$ and differentiating w.r.t. z, $(m - 1)$ times, one gets

$$\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} a_n(m + n)(m + n - 1) \ldots (n + 2)(z - z_0)^{n+1} + b_1 \cdot (m - 1)! \tag{1.82}$$

Therefore,
$$b_1 = \frac{1}{(m - 1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]_{z=z_0} \tag{1.83}$$

3. If z_0 is an essential singularity the above method will not work. But the Laurent expansion can be used directly in this case to get the residue. Thus for the function $f(z) = \exp\left(\frac{1}{z}\right)$, $z = 0$ is an essential singularity. The Laurent expansion around $z = 0$,

$$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \ldots$$

immediately gives $\qquad b_1 = 1.$

4. Sometimes the function $f(z)$ is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$, where $\phi(z)$ and $\psi(z)$ are analytic functions in some domain. If $\psi(z_0) = 0$, then z_0 is a singular point of $f(z)$. If in addition $\phi(z_0) \neq 0$ and $\psi'(z_0) \neq 0$, then z_0 is a simple pole of $f(z)$.

To see this we expand both $\phi(z)$ and $\psi(z)$ around z_0 in Taylor series:

$$\phi(z) = \phi(z_0) + (z - z_0)\phi'(z_0) + \ldots \tag{1.84}$$

$$\psi(z) = (z - z_0)\psi'(z_0) + \frac{(z - z_0)^2}{2!}\psi''(z_0) + \ldots \tag{1.85}$$

Then

$$\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}\frac{(z-z_0)[\phi(z_0)+(z-z_0)\phi'(z_0)+\dots]}{(z-z_0)\left[\psi'(z_0)+\frac{(z-z_0)}{2!}\psi''(z_0)+\dots\right]}$$

$$=\frac{\phi(z_0)}{\psi'(z_0)}$$

is finite. Therefore, by definition z_0 is a simple pole of $f(z)$ and the residue is given by

$$b_1=\frac{\phi(z_0)}{\psi'(z_0)}. \tag{1.86}$$

1.9 Evaluation of Integrals Using Residue Theorem

We now show how the residue theorem can be used to evaluate various types of integrals.

Example 6. Let us evaluate the integral $\oint_C \frac{dz}{\sinh 2z}$, where C is the circle $|z|=2$, the contour being taken in the positive sense.

First we have to find the singularities enclosed by the contour C. The singularities of the integrand are given by

$\sinh 2z=0$ or, $e^{2z}-e^{-2z}=0$, or, $e^{4z}=1=e^{i2\pi n}$, $n=0,\pm1,\pm2,\dots$

or $z=\frac{i\pi n}{2}$.

All the singularities lie along the imaginary axis. The circle $|z|=2$ encloses only the singularities at $z=0,\ \pm\frac{i\pi}{2}$. According to the residue theorem,

$$\oint_C\frac{dz}{\sinh 2z}=2\pi i\left(\text{Residue at }0+\text{Residue at }+\frac{i\pi}{2}+\text{Residue at }-\frac{i\pi}{2}\right) \tag{1.87}$$

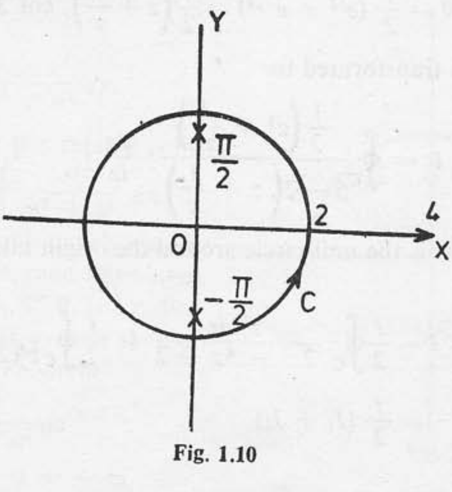

Fig. 1.10

$z = \frac{i\pi}{2}$ is a simple pole of $\frac{1}{\sinh 2z}$, as can be seen by evaluating

$$\lim_{z\to i\pi/2}\left(z-\frac{i\pi}{2}\right)\frac{1}{\sinh 2z} = \lim_{z\to i\pi/2}\left(z-\frac{i\pi}{2}\right)\frac{1}{\left(z-\frac{i\pi}{2}\right)2\cosh 2z\Big|_{i\pi/2}+\cdots}$$

$$= \frac{1}{2\cosh(i\pi)}$$

where, sinh $2z$ in the denominator has been expanded in a Taylor series around $z = \frac{i\pi}{2}$. Since the limit is finite and not equal to zero, $z = \frac{i\pi}{2}$ is a simple pole and the residue at the point $= \frac{1}{2\cosh(i\pi)} = -\frac{1}{2}$. Similarly, $z = 0$ and $z = -\frac{i\pi}{2}$ are also simple poles and the residues at these points are $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. From (1.87),

$$\oint_C \frac{dz}{\sinh 2z} = 2\pi i\left(\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right) = -\pi i \tag{1.88}$$

The residue theorem can also be used to evaluate certain definite integrals involving trigonometric functions. If the integral is of the form $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$ it can be transformed by writing $z = e^{i\theta}$ to the integral $\oint_C f(z)\, dz$, where C is the unit circle $|z| = 1$.

Example 7. Evaluate the integral $I = \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

Let us put $z = e^{i\theta}$. Then

$$dz = ie^{i\theta}\, d\theta, \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right), \cos 3\theta = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$$

The integral is now transformed to

$$I = \oint_C \frac{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)}{5 - 2\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{iz},$$

where the contour C is the unit circle around the origin taken in the positive sense.

$$I = \frac{i}{2}\oint_C \frac{z^3\, dz}{2z^2 - 5z + 2} + \frac{i}{2}\oint_C \frac{dz}{z^3(2z^2 - 5z + 2)}$$

$$= \frac{i}{2}(I_1 + I_2) \tag{1.89}$$

where, $$I_1 = \oint_C \frac{z^3\,dz}{2z^2 - 5z + 2}, \quad \text{and} \quad I_2 = \oint_C \frac{dz}{z^3(2z^2 - 5z + 2)}$$

To evaluate $I_1 = \oint_C \frac{z^3\,dz}{(2z-1)(z-2)}$, we note that the singularities of the integrand are $z = \frac{1}{2}$ and $z = 2$ of which only $z = \frac{1}{2}$ is inside C

$$\lim_{z\to 1/2}\left(z - \frac{1}{2}\right)\frac{z^3}{(2z-1)(z-2)} = -\frac{1}{24}.$$

Thus $z = \frac{1}{2}$ is a simple pole at which the residue is $-1/24$, and

$$I_1 = 2\pi i\left(-\frac{1}{24}\right) = -\frac{\pi i}{12}$$

In I_2 the singularities of the integrand are $z = 0$, $z = \frac{1}{2}$ and $z = 2$. $z = 0$ and $z = \frac{1}{2}$ are inside C. $z = \frac{1}{2}$ is a simple pole, but $z = 0$ is a pole of order 3.

Residue at $z = \dfrac{1}{2}$ is $\lim_{z\to 1/2}\left(z - \dfrac{1}{2}\right)\dfrac{1}{z^3(2z-1)(z-2)} = -\dfrac{8}{3}$ and residue at $z = 0$ is $\lim_{z\to 0}\dfrac{1}{2!}\dfrac{d^2}{dz^2}\dfrac{1}{2z^2 - 5z + 2} = \dfrac{21}{8}$.

Thus $$I_2 = 2\pi i\left(-\frac{8}{3} + \frac{21}{8}\right) = -\frac{\pi i}{12} \text{ and from (1.89)}$$

$$I = \frac{i}{2}(I_1 + I_2) = \frac{\pi}{12}$$

Certain real integrals of the form $\int_{-\infty}^{+\infty} f(x)\,dx$ can also be evaluated by the residue theorem. The method is best explained by an example.

Example 8. Let us evaluate the integral $I = \int_0^\infty \frac{\cos x\,dx}{x^2 + a^2}$, $a > 0$. Since the integrand is an even function x, I can be written as

$$I = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos x}{x^2 + a^2}\,dx.$$

We first evaluate the related complex integral $I_1 = \oint_C \frac{e^{iz}}{z^2 + a^2}\,dz$, where C is the contour shown in Fig. 1.11. The integrand has singularities at $z = \pm ia$. If R, the radius of the semi-circle is greater than a, then by the residue theorem

$$\oint_C \frac{e^{iz}}{z^2 + a^2}\,dz$$

$= 2\pi i$. Residue at $z = +ia$,

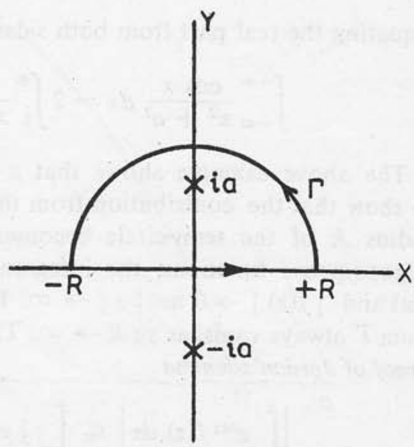

Fig. 1.11

$$= 2\pi i \cdot \left[\frac{e^{iz}}{z+ia}\right]_{z=ia} = \frac{\pi e^{-a}}{a} \tag{1.90}$$

The l.h.s. can be written as

$$\oint_C \frac{e^{iz}}{z^2+a^2}\,dz = \int_{-R}^{+R} \frac{e^{ix}}{x^2+a^2}\,dx + \int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz \tag{1.91}$$

The second integral is the contribution from the semicircular arc Γ, Now,

$$\left|\int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz\right| \leqslant \int_\Gamma \frac{|e^{iz}|\cdot|dz|}{|z^2+a^2|} \tag{1.92}$$

$|e^{iz}| = e^{-y}$. Since Γ is in the upper half plane $y \geqslant 0$. Thus, $|e^{iz}| \leqslant 1$, and

$$\left|\int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz\right| \leqslant \int_\Gamma \frac{|dz|}{|z^2+a^2|} \tag{1.93}$$

On $\quad \Gamma,\ z = Re^{i\theta}, \quad dz = iRe^{i\theta}\,d\theta, \quad |dz| = R\,d\theta.$

Also, $\quad |z^2+a^2| \geqslant |z|^2 - a^2 = R^2 - a^2$

Therefore, one obtains from (1.93),

$$\left|\int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz\right| \leqslant \frac{R}{R^2-a^2}\int_0^\pi d\theta = \frac{\pi R}{R^2-a^2} \tag{1.94}$$

If we now make the radius R of the semi-circle arbitrarily large, $R \to \infty$, and $\left|\int_\Gamma \frac{e^{iz}}{z^2+a^2}\,dz\right| \to 0.$

Then from (1.90) and (1.91),

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi}{a}\,e^{-a} \tag{1.95}$$

Equating the real part from both sides, one obtains the desired integral

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\,dx = 2\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi}{a}\,e^{-a} \tag{1.96}$$

The above example shows that a crucial step involved in the method is to show that the contribution from the semi-circular arc Γ vanishes as the radius R of the semi-circle becomes infinite. In many integrals involving trigonometric functions, the integrand is of the form $e^{iaz} f(z)$, where a is real and $|f(z)| \to 0$ as $|z| \to \infty$. For such integrands, the contribution from Γ always vanishes as $R \to \infty$. This result is known as *Jordon's lemma.*

Proof of Jordon's lemma

$$\left|\int_\Gamma e^{iaz} f(z)\,dz\right| \leqslant \int_\Gamma |e^{iaz}|\cdot|f(z)|\cdot|dz|$$

On, Γ (Fig. 1.11), $z = Re^{i\theta} = R\cos\theta + iR\sin\theta, \qquad dz = iRe^{i\theta}\,d\theta$
Thus,

$$\left|\int_\Gamma e^{iaz} f(z)\,dz\right| \leqslant \int_0^\pi e^{-aR\sin\theta}\,|f(Re^{i\theta})|\,R\,d\theta \tag{1.97}$$

Since, $|f(z)| \to 0$ as $|z| \to \infty$, we can choose R large enough so that $|f(Re^{i\theta})| < \epsilon$, a preassigned positive quantity. Then,

$$\left|\int_\Gamma e^{iaz} f(z)\,dz\right| \leqslant \epsilon R\int_0^\pi e^{-aR\sin\theta}\,d\theta \tag{1.98}$$

Using the result that $\int_0^{\pi/2} e^{-aR\sin\theta}d\theta = \int_{\pi/2}^{\pi} e^{-aR\sin\theta}\,d\theta$

we can rewrite (1.98) as

$$\left|\int_\Gamma e^{iaz} f(z)\,dz\right| \leqslant 2\epsilon R\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta \tag{1.99}$$

From the plot of $y = \sin\theta$ and $y = 2\theta/\pi$ (Fig. 1.12), it is clear that in the range $0 \leqslant \theta \leqslant \pi/2$, $\sin\theta \geqslant 2\theta/\pi$. Therefore,

$$\left|\int_\Gamma e^{iaz} f(z)\,dz\right| \leqslant 2\epsilon R\int_0^{\pi/2} e^{-aR\,\frac{2\theta}{\pi}}\,d\theta$$

$$= 2\epsilon R\cdot\frac{\pi}{2aR}\left[1 - e^{-aR}\right] \tag{1.100}$$

As $R \to \infty$, $\epsilon \to 0$, $\qquad \lim_{R\to\infty}\left|\int_\Gamma e^{iaz} f(z)\,dz\right| \to 0.$

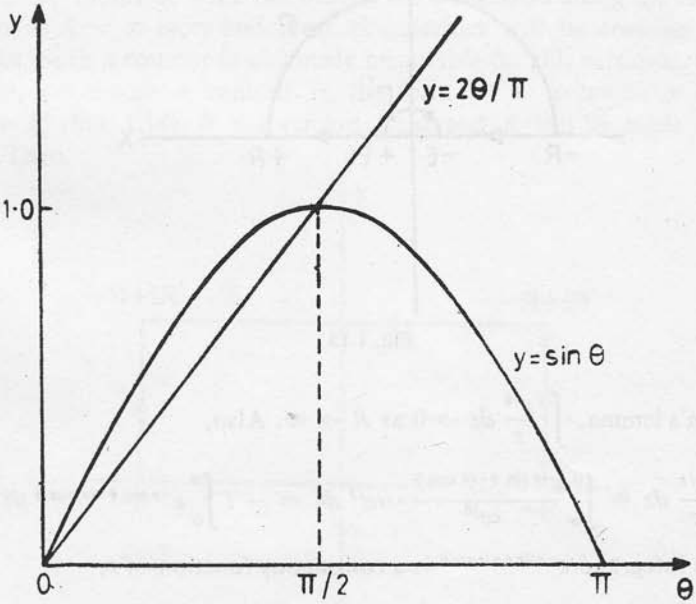

Fig. 1.12

Example 9. Jordon's lemma can be used in the evaluation of the integral $\int_0^\infty \frac{\sin x}{x}\,dx$. As before, we first convert the integral to an integral from $-\infty$ to $+\infty$, thus:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$$

and evaluate the related complex integral

$$\oint_C \frac{e^{iz}}{z}\,dz.$$

In choosing the contour C in this case, we have to be careful because the singularity, $z = 0$ of the integrand falls on the real axis. If the contour is allowed to pass through this point the residue theorem will not hold. To avoid this difficulty, the contour near $z = 0$ is bent in the form of a semicircle Γ' of radius ϵ as shown in Fig. 1.13. Then,

$$\oint_C \frac{e^{iz}}{z}\,dz = 0$$

or,
$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{+\epsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{\Gamma'} \frac{e^{iz}}{z}\,dz + \int_{\Gamma} \frac{e^{iz}}{z}\,dz = 0 \qquad (1.101)$$

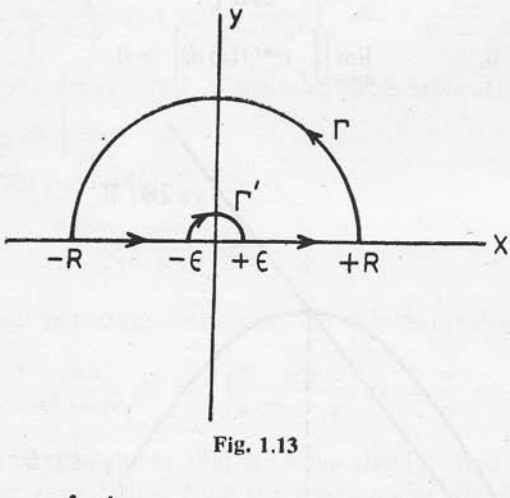

Fig. 1.13

By Jordon's lemma, $\int_\Gamma \frac{e^{iz}}{z}\,dz \to 0$ as $R \to \infty$. Also,

$$\int_{\Gamma'} \frac{e^{iz}}{z}\,dz = \int_\pi^0 \frac{e^{-\epsilon \sin\theta + i\epsilon\cos\theta}}{\epsilon e^{i\theta}}\cdot i\epsilon e^{i\theta}\,d\theta = -i\int_0^\pi e^{-\epsilon\sin\theta + i\epsilon\cos\theta}\,d\theta$$

Since, the integrand $e^{-\epsilon\sin\theta + i\epsilon\cos\theta}$ is a continuous function of ϵ,

$$\lim_{\epsilon\to 0} \int_{\Gamma'} \frac{e^{iz}}{z}\,dz = -i \lim_{\epsilon\to 0}\int_0^\pi e^{-\epsilon\sin\theta + i\epsilon\cos\theta}\,d\theta = -i\pi \qquad (1.102)$$

Thus, with $R \to \infty$ and $\epsilon \to 0$, we obtain from (1.101)

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\, dx = i\pi \tag{1.103}$$

Equating the imaginary part from both sides, one gets

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = 2\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi \tag{1.104}$$

In the above example, the semi-circle Γ' was drawn in the upper half plane so that the singularity at the origin was outside the contour. Alternatively Γ' could also be drawn in the lower half plane taking the singularity inside. The value of the integral obtained would of course be the same. This can be easily verified and is left as an exercise.

In all the examples of evaluation of real integrals, the contour has been chosen in the form of a semicircle. But this need not be the case. In some problems the choice of a semi-circle with its base on the real axis as the contour is unsuitable as is illustrated in the next example.

Example 10. To evaluate $I = \int_{-\infty}^{+\infty} \frac{x^2 e^x}{e^{2x}+1}\, dx$, we calculate the related complex integral $I_1 = \oint_C \frac{z^2 e^{iz}}{e^{2z}+1}\, dz$, where the contour C is yet to be specified. The singularities of the integrand are given by $c^{2z} = -1 = e^{\mp i(2n+1)\pi}$, $n = 0, 1, 2, \ldots$, or $z = \pm \frac{i\pi}{2}, \pm \frac{3i\pi}{2}, \ldots$. The singularities all lie along the imaginary axis and they are infinite in number. If a semi-circular contour of radius R with the base of the semi-circle along the real axis is chosen, as $R \to \infty$ more and more singularities will be crossing into the contour. Such a contour is obviously unsuitable for this particular problem. Instead, we choose a contour in the form of a rectangle $x = \pm R$, $y = + R'$ (Fig. 1.14). R' will remain fixed and R will be made infinitely large. Then,

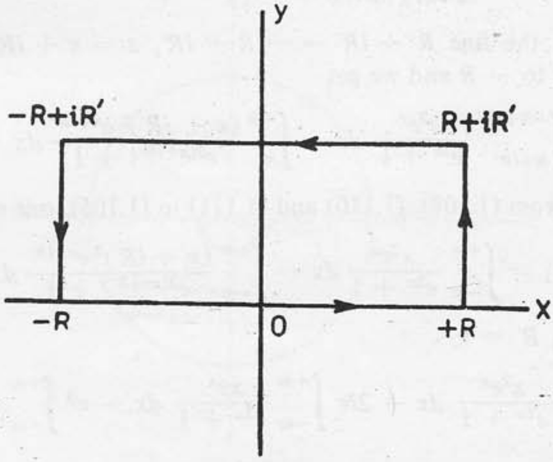

Fig. 1.14

$$I_1 = \int_{-R}^{+R} \frac{x^2 e^x}{e^{2x}+1}\,dx + \int_{R}^{R+iR'} \frac{z^2 e^z}{e^{2z}+1}\,dz + \int_{R+iR'}^{-R+iR'} \frac{z^2 e^z}{e^{2z}+1}\,dz$$

$$+ \int_{-R+iR'}^{-R} \frac{z^2 e^z}{e^{2z}+1}\,dz \tag{1.105}$$

Now,

$$\left| \int_{R}^{R+iR'} \frac{z^2 e^z}{e^{2z}+1}\,dz \right| \leqslant \int_{R}^{R+iR'} \frac{|z^2| \cdot |e^z| \cdot |dz|}{|e^{2z}+1|} \leqslant \int_{R}^{R+iR'} \frac{|z^2| \cdot |e^z|\,|dz|}{|e^{2z}|-1} \tag{1.106}$$

On the line $R \to R + iR'$, $z = R + iy$, $dz = idy$ and y varies from 0 to R'. Also $|z| = \sqrt{R^2 + y^2}$. From (1.106), we get

$$\left| \int_{R}^{R+iR'} \frac{z^2 e^z}{e^{2z}+1}\,dz \right| \leqslant \int_{0}^{R'} \frac{(R^2+y^2)\,e^R\,dy}{e^{2R}-1}$$

$$\leqslant \int_{0}^{R'} \frac{(R^2+R'^2)\,e^R}{e^{2R}-1}\,dy$$

$$= \frac{(R^2+R'^2)\,e^R \cdot R'}{e^{2R}-1} \tag{1.107}$$

Therefore,

$$\lim_{R\to\infty} \left| \int_{R}^{R+iR'} \frac{z^2 e^z}{e^{2z}+1}\,dz \right| = 0 \tag{1.108}$$

Similarly, on the line $-R + iR' \to -R$, $z = -R + iy$, $dz = i\,dy$ y varies from R' to zero. Hence,

$$\left| \int_{-R+iR'}^{-R} \frac{z^2 e^z}{e^{2z}+1}\,dz \right| \leqslant \int_{R'}^{0} \frac{(R^2+y^2)\,e^{-R}}{e^{-2R}-1}\,dy \leqslant \frac{(R^2+R'^2)\,e^{-R}\,R'}{1-e^{-2R}} \tag{1.109}$$

Thus,

$$\lim_{R\to\infty} \left| \int_{-R+iR'}^{-R} \frac{z^2 e^z}{e^{2z}+1}\,dz \right| = 0 \tag{1.110}$$

Finally, on the line $R + iR' \to -R + iR'$, $z = x + iR'$, $dz = dx$, x varies from R to $-R$ and we get

$$\int_{R+iR'}^{-R+iR'} \frac{z^2 e^z}{e^{2z}+1}\,dz = \int_{R}^{-R} \frac{(x+iR')^2\,e^{x+iR'}}{e^{2(x+iR')}+1}\,dx \tag{1.111}$$

Substituting from (1.108), (1.110) and (1.111) in (1.105), one obtains

$$I_1 = \int_{-\infty}^{+\infty} \frac{x^2 e^x}{e^{2x}+1}\,dx - \int_{-\infty}^{+\infty} \frac{(x+iR')^2\,e^{x+iR'}}{e^{2(x+iR')}+1}\,dx \tag{1.112}$$

If we now let, $R' = \pi$

$$I_1 = 2\int_{-\infty}^{+\infty} \frac{x^2 e^x}{e^{2x}+1}\,dx + 2i\pi \int_{-\infty}^{+\infty} \frac{xe^x}{e^{2x}+1}\,dx - \pi^2 \int_{-\infty}^{+\infty} \frac{e^x}{e^{2x}+1}\,dx \tag{1.113}$$

In the second integral on the r.h.s., the integrand $\dfrac{x}{e^x + e^{-x}}$ is an odd function of x. Therefore the integral vanishes.

Also, $$\int_{-\infty}^{+\infty} \frac{e^x}{e^{2x}+1}\,dx = \int_0^{\infty} \frac{du}{u^2+1} = \frac{\pi}{2}$$

Thus, $$I_1 = 2\int_{-\infty}^{+\infty} \frac{x^2 e^x}{e^{2x}+1}\,dx - \frac{\pi^3}{2} \tag{1.114}$$

On the other hand, $I_1 = \oint_C \dfrac{z^2 e^z}{e^{2z}+1}\,dz = 2\pi i \left(\text{Residue at } \dfrac{i\pi}{2}\right)$, $\dfrac{i\pi}{2}$ being the only singularity enclosed by the contour. The integrand is of the form $\dfrac{\phi(z)}{\psi(z)}$, $\phi\left(\dfrac{i\pi}{2}\right) \neq 0$, $\psi\left(\dfrac{i\pi}{2}\right) = 0$ and $\psi'\left(\dfrac{i\pi}{2}\right) \neq 0$. Therefore the residue

$$= \frac{\phi(i\pi/2)}{\psi'(i\pi/2)} = -\frac{\pi^2/4\cdot i}{2(-1)} = \frac{\pi^2}{8}\,i.$$

Thus, $$I_1 = 2\pi i\left(\frac{\pi^2}{8}\,i\right) = -\frac{\pi^3}{4} \tag{1.115}$$

From (1.114) and (1.115), we get $$\int_{-\infty}^{+\infty} \frac{x^2 e^x}{e^{2x}+1}\,dx = \frac{\pi^3}{8} \tag{1.116}$$

The next example involves, integration around a branch point.

Example 11. $I = \displaystyle\int_0^{\infty} \frac{x^{-\alpha}}{1+x}\,dx, \qquad 0 < \alpha < 1.$

We evaluate the related complex integral,

$$I_1 = \oint_C \frac{z^{-\alpha}}{1+z}\,dz$$

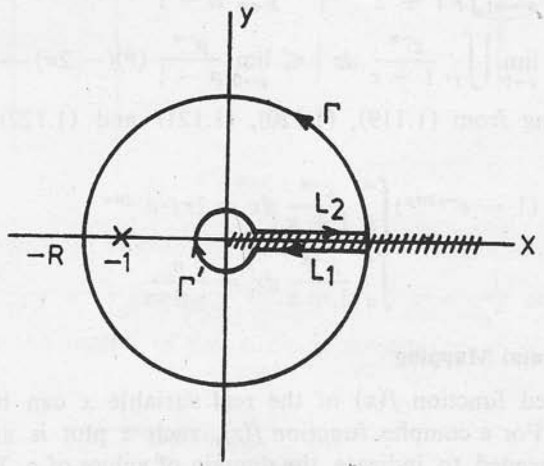

Fig. 1.15

With $0 < \alpha < 1$, the integrand of I_1 has a branch point singularity at $z = 0$.

$$z^{-\alpha} = e^{-\alpha \log_e \gamma - \alpha i\theta} \equiv e^{-\alpha \log_e \gamma - \alpha i\theta} = |z|^{-\alpha} e^{-\alpha i\theta}$$

If the domain of argument θ is restricted as $0 \leqslant \theta < 2\pi$, the branch cut is along the positive real axis. It is not possible, therefore, to choose a contour along the real axis. The integrand has also a simple pole at $z = -1$.

A contour C as shown in Fig. 1.15 can be chosen in this case. This consists of two lines L_1 and L_2 running parallel to the branch cut, part of a circle Γ' of radius ρ, and part of a circle Γ of radius $R > 1$. In the limit the straight lines will approach the real axis from above and below, ρ will tend to zero and R will be made arbitrarily large.

$$\oint_C \frac{z^{-\alpha}}{1+z}\, dz = 2\pi i. \quad \text{(Residue at } z = -1)$$

$$= 2\pi i \cdot e^{-i\pi\alpha} \tag{1.117}$$

or, $$\int_{L_1} \frac{z^{-\alpha}}{1+z}\, dz + \int_{\Gamma'} \frac{z^{-\alpha}}{1+z}\, dz + \int_{L_2} \frac{z^{-\alpha}}{1+z}\, dz + \int_{\Gamma} \frac{z^{-\alpha}}{1+z}\, dz$$
$$= 2\pi i \cdot e^{-i\pi\alpha} \tag{1.118}$$

Along L_1 in the limit when the line approaches the real axis.

$$\int_{L_1} \frac{z^{-\alpha}}{1+z}\, dz = \int_\infty^0 \frac{|z|^{-\alpha} e^{-\alpha i 2\pi}}{1+z}\, dz = -\, e^{-i2\pi\alpha} \int_0^\infty \frac{x^{-\alpha}}{1+x} dx \tag{1.119}$$

Similarly, $$\int_{L_2} \frac{z^{-\alpha}}{1+z}\, dz = \int_0^\infty \frac{x^{-\alpha}}{1+x} dx \tag{1.120}$$

Also, $$\left| \int_\Gamma \frac{z^{-\alpha}}{1+z}\, dz \right| \leqslant \int_\Gamma \frac{|z|^{-\alpha}}{|1+z|} |dz| \leqslant \int \frac{|z|^{-\alpha}}{|z|-1} |dz|,$$

On Γ, $z = Re^{i\theta}$, $dz = Re^{i\theta} i\, d\theta$. Therefore,

$$\lim_{R\to\infty} \left| \int_\Gamma \frac{z^{-\alpha}}{1+z}\, dz \right| \leqslant \lim_{R\to\infty} \frac{R^{-\alpha}}{R-1} \cdot R\, 2\pi \to 0 \tag{1.121}$$

and $$\lim_{\rho\to 0} \left| \int_{\Gamma'} \frac{z^{-\alpha}}{1+z}\, dz \right| \leqslant \lim_{\rho\to 0} \frac{\rho^{-\alpha}}{\rho-1} (\rho)(-2\pi) \to 0 \tag{1.122}$$

Substituting from (1.119), (1.120), (1.121) and (1.122) in (1.118), we obtain

$$(1 - e^{-i2\pi\alpha}) \int_0^\infty \frac{x^{-\alpha}}{1+x}\, dx = 2\pi i \cdot e^{-i\pi\alpha}$$

or, $$\int_0^\infty \frac{x^{-\alpha}}{1+x}\, dx = \frac{\pi}{\sin \pi\alpha} \tag{1.123}$$

1.10 Conformal Mapping

A real valued function $f(x)$ of the real variable x can be plotted in the y-direction. For a complex function $f(z)$, such a plot is not possible. The z-plane is needed to indicate the domain of values of z. The function $f(z)$

$= w = u + iv$ can however be plotted in a separate plane, the so called "w-plane" in which u and v are plotted along two perpendicular axes. One can consider this a transformation in which a set of points in the z-plane is mapped into a set of points in the w-plane.

As an example consider the mapping $w = \frac{1}{z}$. We have $u + iv = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ so that $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$.
These equations can also be inverted to write x and y in terms of u and v. The easiest way to do this is to note that $z = 1/w$. Thus,

$$x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = -\frac{v}{u^2+v^2} \tag{1.124}$$

From the above equations we can easily draw the following conclusions:

(1) A horizontal line $y = c_1$ in the z-plane will be transformed into the curve, $-\frac{v}{u^2+v^2} = c_1$ or, $u^2 + \left(v + \frac{1}{2c_1}\right)^2 = \left(\frac{1}{2c_1}\right)^2$ in the w-plane. This is a circle of radius $1/2c_1$ and centre at $(0, -1/2c_1)$.

(2) A vertical line $x = c_2$ in the z-plane will be transformed into the circle $\frac{u}{u^2+v^2} = c_2$ or, $\left(u - \frac{1}{2c_2}\right)^2 + v^2 = \left(\frac{1}{2c_2}\right)^2$.

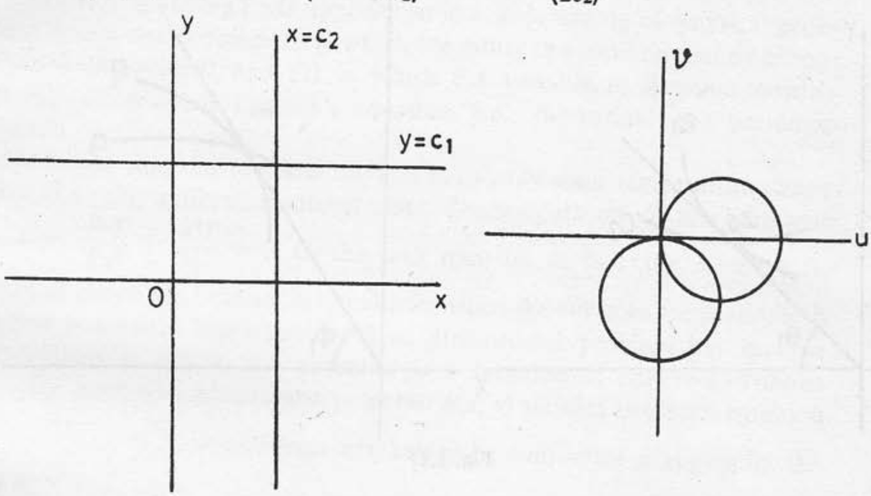

Fig. 1.16

(3) A circle $x^2 + y^2 = r^2$ in the z-plane will be transformed into another circle $\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2$ or, $u^2 + v^2 = \frac{1}{r^2}$ is the w-plane. For a point in the interior of the circle in the z-plane, $x^2 + y^2 < r^2$. This gives,

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} < r^2$$

or, $\dfrac{1}{u^2+v^2} < r^2$ or $u^2+v^2 > \dfrac{1}{r^2}$.

Thus the interior of the circle in the z-plane is mapped into the exterior of the circle in the w-plane. Similarly, the exterior of the circle in the z-plane is mapped into the interior of the circle in the w-plane.

Definition

A mapping is said to be conformal at a point if the angle between two curves passing through the point is preserved in magnitude and sense by the mapping.

Suppose by a mapping $w = f(z)$, the curves C_1 and C_2 in the z-plane are mapped into the curves Γ_1 and Γ_2 in the w-plane. The point of intersection z_0 of C_1 and C_2 is mapped into the point of intersection w_0 of Γ_1 and Γ_2.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}\bigg|_{z_0}$$

$$\arg f'(z) = \arg\left(\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}\bigg|_{z_0}\right)$$

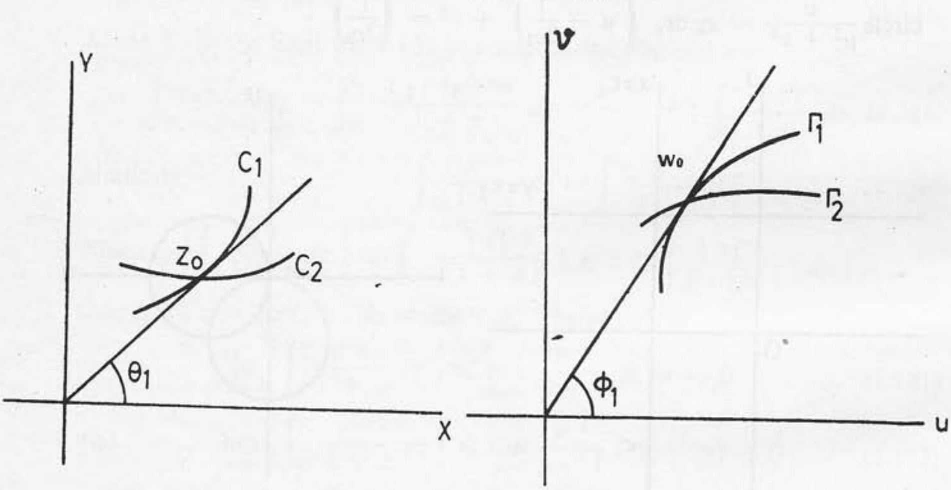

Fig. 1.17

If the limit exists, then

$$\arg f'(z_0) = \lim_{\Delta w \to 0} (\arg \Delta w - \arg \Delta z)\Big|_{z_0}$$
$$= \lim_{\Delta z \to 0} \arg \Delta w\Big|_{w_0} - \lim_{\Delta z \to 0} \arg \Delta z\Big|_{z_0} \tag{1.125}$$

If both z_0 and $z_0 + \Delta z$ lie on C_1, then $\lim_{\Delta z \to 0} \arg \Delta z\Big|_{z_0} = \theta_1$, the angle the tangent to C_1 at z_0 makes with the x-axis and $\lim_{\Delta w \to 0} \arg \Delta w\Big|_{w_0} = \phi_1$ the angle of inclination of the tangent to Γ_1 at w_0 with the u-axis. Thus, if $f'(z_0)$ exists, then

$$\phi_1 - \theta_1 = \arg f'(z_0) \tag{1.126}$$

Similarly, approaching z_0 along C_2

$$\phi_2 \quad \theta_2 = \arg f'(z_0) \tag{1.127}$$

where, θ_2 = the angle the tangent at z_0 to C_2 makes with the x-axis, and
ϕ_2 = the angle the tangent at w_0 to Γ_2 makes with the u-axis.
From (1.126) and (1.127), we get

$$\theta_2 - \theta_1 = \phi_2 - \phi_1 \tag{1.128}$$

But $(\theta_2 - \theta_1)$ is the angle between the curves C_1 and C_2 at z_0 and $(\phi_2 - \phi_1)$ is the angle between the curves Γ_1 and Γ_2 at w_0. Therefore, by the above mapping the angle between the curves C_1 and C_2 is preserved in magnitude and sense. The mapping by definition, is *conformal.*

In arriving at the above result, we have assumed that $f'(z_0)$ exists. Also, if $f'(z_0)$ exists but is equal to zero its argument is undefined and the above reasoning will not go through. Therefore the necessary conditions for a mapping, $w = f(z)$ to be conformal at z_0 are, (1) $f'(z_0)$ must exist and, (2) $f'(z_0) \neq 0$. If the mapping is to be conformal over a domain, $f'(z)$ must exist at all points of the domain and condition (1) is equivalent to $f(z)$ being analytic in the domain.

Conformal mapping finds application in a wide variety of physical problems. These are problems, (1) which are either two dimensional or effectively two dimensional, and (2) in which the variable in question satisfies the two dimensional Laplace's equation, i.e., the variable is a harmonic function.

Thus, one may be required to find the steady state temperature at any point of a thin, uniform, insulated plate. The temperature T is a harmonic function $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Or the task may be to find the electrostatic potential inside an infinite half cylinder when the surfaces are maintained at given potentials. The later is a three dimensional problem but because the cylinder is infinite the potential is a function of only two variables (x and y, say). The electrostatic potential $\phi(x, y)$ satisfies Laplace's equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. Such problems are solved by conformal mapping by the following procedure:

First the configuration in the z-plane is mapped into another in the w-plane by a conformal mapping. The mapping is so chosen that the resulting problem is a simple one whose solution can be obtained easily. By the inverse mapping, one then transforms back the solution in the w-plane to obtain the solution of the original problem. The prescription just given is based on rigorous mathematical reasoning. We state without proof the following theorems.

Theorem

If the mapping $f(z) = u(x, y) + iv(x, y)$ is conformal over a domain in the z-plane, and $H(u, v)$ is a harmonic function in the w-plane, then the function

$\phi(x, y) = H(u(x, y), v(x, y))$ obtained by transforming $H(u, v)$ to the z-plane is also a harmonic function in the given domain.

If, $\dfrac{\partial^2 H}{\partial u^2} + \dfrac{\partial^2 H}{\partial v^2} = 0$, then the transformed function satisfies $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$. Thus if we have found a solution of Laplace's equation in the w-plane, then by transforming back by a conformal mapping, we obtain a solution of Laplace's equation in the z-plane.

Theorem

Let the curve C in the z-plane be mapped into the curve Γ in the w-plane by the conformal mapping $f(z) = u(x, y) + iv(x, y)$. If $H(u, v)$ is a harmonic function in the w-plane satisfying either of the boundary conditions $H(u, v) = c_1$ on Γ or $dH/dn = 0$ on Γ (d/dn is the derivative normal to Γ), then function $\phi(x, y) = H(u(x, y), v(x, y))$ also satisfies the same boundary conditions on C.

Thus, if a solution of Laplace's equation satisfying boundary conditions as given above, is found in the w-plane, then by transforming back by a conformal mapping one obtains a solution of Laplace's equation in the z-plane satisfying corresponding boundary conditions.

In the following, we solve several problems using conformal mapping.

Example 12. Two co-axial infinite cylinders of radii ρ_1 and ρ_2 ($\rho_1 < \rho_2$) are maintained at potentials V_1 and V_2 respectively. We are to find the electrostatic potential at any point between the two cylinders.

If the axis of the cylinders is in the z-direction, the potential ϕ is a function of x and y only. $\phi = V_1$ on the surface $x^2 + y^2 = \rho_1^2$ and $\phi = V_2$ on $x^2 + y^2 = \rho_2^2$. By the conformal mapping $w = \log_e z$, ($z \neq 0$) the circles in the z-plane are mapped into two infinite parallel lines in the w-plane.

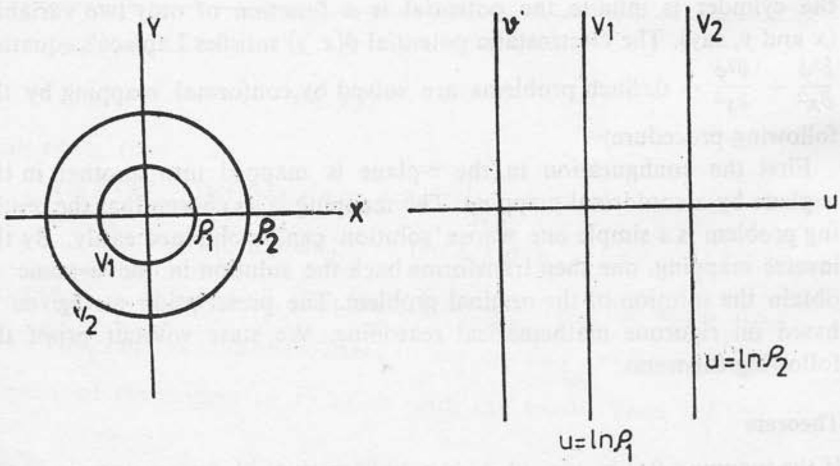

Fig. 1.18

Thus,

$$z = e^w = e^{u+iv}$$

or
$$x + iy = e^u \cos v + ie^u \sin v$$

Thus,
$$x = e^u \cos v, \quad \text{and} \quad y = e^u \sin v \tag{1.129}$$
$$x^2 + y^2 = (e^u)^2$$

The circle $x^2 + y^2 = \rho_1^2$ is mapped into the line $u = \log_e \rho_1$ and $x^2 + y^2 = \rho_2^2$ is mapped into $u = \log_e \rho_2$. The problem in the w-plane now is to find the harmonic function (the potential), $H(u, v)$ between the infinite parallel lines such that $H = V_1$ for $u = \log_e \rho_1$, and $H = V_2$ for $u = \log_e \rho_2$. The solution is given by,

$$H(u, v) = \frac{V_2 - V_1}{\log_e \rho_2 - \log_e \rho_1}(u - \log_e \rho_1) + V_1 \tag{1.130}$$

The solution to the original problem is obtained by transforming back to the z-plane by putting $u = \log_e \sqrt{x^2 + y^2} = \log_e r$. After simplification, one gets

$$\phi(x, y) = V_2 \frac{\log_e\left(\frac{r}{\rho_1}\right)}{\log_e\left(\frac{\rho_2}{\rho_1}\right)} + V_1 \frac{\log_e\left(\frac{\rho_2}{r}\right)}{\log_e\left(\frac{\rho_2}{\rho_1}\right)} \tag{1.131}$$

Example 13. The same mapping can be used to find the potential on the upper half of the z-plane if the positive half of the x-axis is maintained at potential V_0 and the negative half at $-V_0$. $w = \log_e z = \log_e r + i\theta$, $-\pi < \theta \leqslant \pi$, $u = \log_e r, v = \theta$.

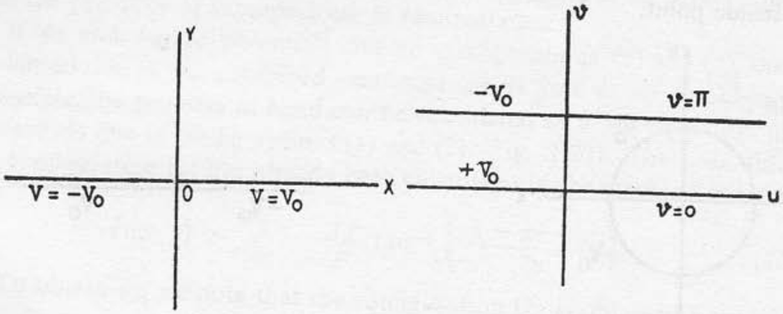

Fig. 1,19

The positive half of the x-axis which corresponds to $\theta = 0$, is mapped into the line $v = 0$, and the negative half of the x-axis (approaching from above) is mapped into the line $v = \pi$. In the w-plane the potential between the lines is given by,

$$H(u, V) = V_0 - \frac{2V_0}{\pi} v \tag{1.132}$$

Thus
$$\phi(x, y) = V_0 - \frac{2V_0}{\pi} \tan^{-1} \frac{y}{x} \tag{1.133}$$

Example 14. A solid in the form of an infinite cylindrical wedge has its surfaces at $\theta = 0$ and $\theta = \theta_0$ maintained at temperatures 0° and T_0° and the curved surface $r = r_0$ is insulated (Fig. 1.20). To find the temperature at

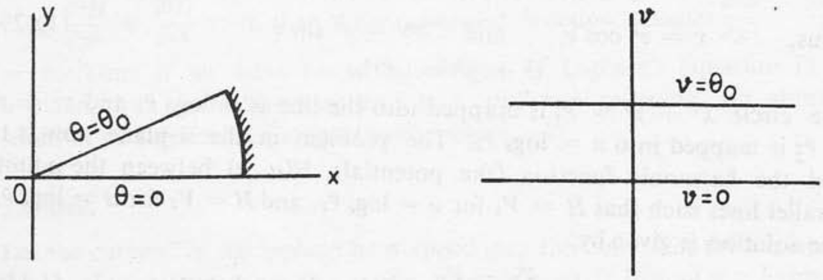

Fig. 1.20

any point inside, we can again use the mapping $w = \log_e z$. The surfaces $\theta = 0$ and $\theta = \theta_0$ are mapped into lines, $v = 0$ and $v = \theta_0$. The solution of the problem in the w-plane is given by $H(u, v) = \dfrac{T_0^\circ}{\theta_0}\, v$ and the required temperature, therefore is,

$$T = \frac{T_0^\circ}{\theta_0} \tan^{-1} \frac{y}{x}$$

Example 15. Now consider the problem of two infinite half cylinders of unit radius with the upper half maintained at potential $+ V_0$ and the lower half maintained at constant potential $- V_0$. We are to find the potential at any inside point.

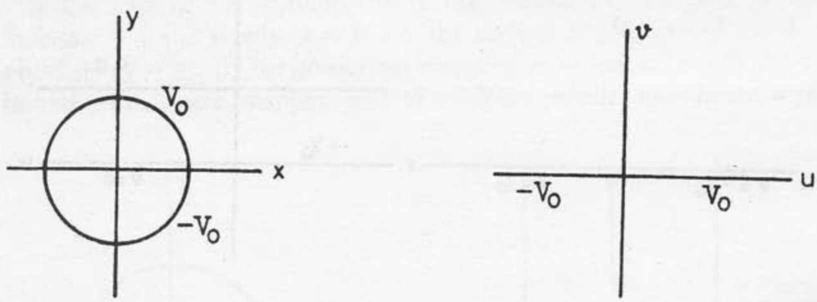

Fig. 1.21

By the transformation $z = \dfrac{i - w}{i + w}$, the interior of the unit circle is mapped into the upper half of the w-plane. Inverting the relation, we obtain

$$\begin{aligned} w &= i\,\frac{1 - z}{1 + z} = i\,\frac{(1 - x) - iy}{(1 + x) + iy} \\ &= i\,\frac{1 - x^2 - y^2}{(1 + x)^2 + y^2} + \frac{2y}{(1 + x)^2 + y^2} \\ &= u + iv \end{aligned}$$

$$u = \frac{2y}{(1+x)^2+y^2} \quad \text{and} \quad v = \frac{1-x^2-y^2}{(1+x)^2+y^2} \tag{1.134}$$

If, $\quad x^2 + y^2 \leqslant 1. \quad v \geqslant 0.$

Thus, the interior of the circle is mapped into the upper half of the w-plane. If $x^2 + y^2 = 1$, and $y > 0$, $v = 0$ and $u > 0$. The upper half of the circle is therefore mapped into the positive u-axis. Similarly, the lower half of the circle is mapped into the negative u-axis. The problem in the w-plane is thus to find a harmonic function $H(u, v)$ in the upper half of the w-plane, which has the value $+ V_0$ on the positive u-axis and the value $- V_0$ on the negative u-axis. This has already been solved (but in the z-plane) in Example 13. From (1.133), $H(u, v) = V_0 - \frac{2V_0}{\pi} \tan^{-1} \frac{v}{u}$. Therefore the required potential $\phi(x, y)$ inside the cylinder is given by

$$\phi(x, y) = V_0 - \frac{2V_0}{\pi} \tan^{-1}\left(\frac{1-x^2-y^2}{2y}\right) \tag{1.135}$$

Finally, we show in the next example how the principle of superposition combined with conformal mapping can be used to solve certain problems.

Example 16. An infinite hollow cylinder of unit radius and axis along the z-axis is cut into four equal parts by the planes $x = 0$, $y = 0$. The segments in the first and third quadrant are maintained at potential $+ V_0$ and $- V_0$ respectively, and the segments in the second and fourth quadrant are maintained at zero potential. To find the potential at an inside point, we use the principle of superposition in electrostatics:

If ϕ_1 and ϕ_2 are potentials due to configurations (1) and (2), then the potential due to the combined configuration is just $\phi_1 + \phi_2$. Using this principle, the problem at hand can be considered as a superposition of two potentials due to configurations (1) and (2) (Fig. 1.22). The potential due to configuration (1) has already been calculated in the previous example, as

$$\phi_1(x, y) = \frac{V_0}{2} - \frac{V_0}{\pi} \tan^{-1}\left(\frac{1-x^2-y}{2y}\right) \tag{1.136}$$

To obtain ϕ_2, we note that the configuration (2) is obtained from (1) by

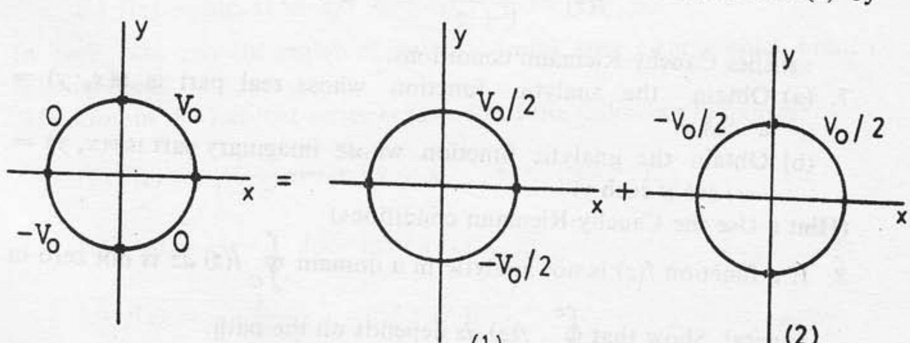

Fig. 1.22

its rotation by $\pi/2$ about the z-azis in the clockwise direction. As a result of this transformation $y \rightarrow x$ and $x \rightarrow -y$,

$$\phi_2(x, y) = \frac{V_0}{2} - \frac{V_0}{\pi} \tan^{-1}\left(\frac{1 - x^2 - y^2}{2x}\right) \tag{1.137}$$

And the potential $\phi(x, y)$ at any inside point is given by

$$\phi(x, y) = V_0 - \frac{V_0}{\pi} \tan^{-1}\left(\frac{1 - x^2 - y^2}{2y}\right)$$
$$- \frac{V_0}{\pi} \tan^{-1}\left(\frac{1 - x^2 - y^2}{2x}\right) \tag{1.138}$$

Conformal mapping is a convenient method of solving certain problems. But its usefulness is limited by the following facts:

(1) Only the solution of two dimenional Laplace's equation can be obtained by conformal mapping. The problem must either be two dimensional or effectively two dimensional.

(2) Knowledge of a lot of mappings is required to find a suitable mapping for a given problem.

EXERCISE-I

1. Prove that if $z_1 z_2 = 0$ than at least one of the factors is zero.
2. Prove that $||z_1| - |z_2|| \leqslant |z_1 - z_2| \leqslant |z_1| + |z_2|$
3. In Example 2, it was shown that the function $f(z) = |z|^2$ is differentiable only at the point $z = 0$. Examine the continuity of the function in the z-plane.
4. Determine whether the tunctions, (*a*) $f(z) = u(x, y)$, (b) $f(z) = iv(x, y)$ are differentiable anywhere in the z-plane, where u and v are real functions.
5. Obtain the Cauchy-Riemann conditions in polar co-ordinates.
6. Determine whether the function

$$f(z) = \frac{z^*}{1 + z}, \qquad z \neq -1$$

satisfies Cauchy-Riemann conditions.

7. (a) Obtain the analytic function whose real part is $u(x, y) = e^x \cos y$.

   (b) Obtain the analytic function whose imaginary part is $v(x, y) = -i \cos x \cosh y$.

(**Hint :** Use the Cauchy-Riemann conditions)

8. If a function $f(z)$ is not analytic in a domain $\oint_C f(z)\, dz$ is not zero in general. Show that $\oint_a^b f(z)\, dz$ depends on the path.

9. Show that if $f(z)$ is analytic in some domain then the integral $\oint_a^b f(z)dz$ is path independent.
10. Choosing suitable paths evaluate the integrals:

(a) $\int_1^i e^z\, dz$ (b) $\int_{1-i}^{1+i} z^2\, dz$ (c) $\int_0^{1+i} [(x^2 + xy) + i(y^2 + xy)]\, dz$

11. Show that $\oint_C z^n\, dz = 0, \quad \text{if } n \neq -1$

$$= 2\pi i, \quad \text{if} \quad n = -1$$

where, n is an integer and C is a circle around $z = 0$

12. Prove that if $f(z)$ is analytic and not constant in the interior of a region, then $|f(z)|$ has no maximum value in that interior.
13. Prove the fundamental theorem of algebra: any polynomial of degree $n \geqslant 1$ has at least one zero. Hence prove that for $n \geqslant 1$, a polynomial of degree n has no more than n distinct zeros. (Hint: apply Liouville's Theorem to the inverse of the polynomial).
14. Use Cauchy integral formula to evaluate the following integrals:

(a) $\oint_C \frac{z}{2z+1}\, dz$ (b) $\oint_C \frac{\cos z}{z(z^2+9)}\, dz$

(c) $\oint_C \frac{\sin z}{dz}\, dz$ (d) $\oint_C \frac{\sin z}{\left(z - \frac{\pi}{4}\right)^2}\, dz$

where, C is the circle $|z| = 2$.

15. Expand the following functions in Taylor series:

(a) $f(z) = \frac{1}{\sqrt{1+z}}$ around $z = 0$

(b) $f(z) = \frac{1}{1-z}$ around $z = i$

(c) $f(z) = \frac{z-1}{z^2}$ around $z = 1$

(d) $f(z) = \sinh z$ around $z = i\pi$

(e) $f(z) = \log_e (1 + z)$ around $z = 0$

In each case give the region of validity. In (e), state clearly which branch is being used.

16. Obtain the Laurent series expansion of the following functions:

(a) $f(z) = \frac{z^4}{1+z}$ around $z = 0$ for $|z| > 1$.

(b) $f(z) = \frac{1}{1-z}$ for $|z - i| > \sqrt{2}$

(c) $f(z) = \frac{1}{z^2 \sinh z}$ around $z = 0$

(d) $f(z) = \dfrac{1}{z^2 - 5z + 6}$ for $2 < z < 3$.

17. Evaluate the following integrals using the residue theorem:

(a) $\oint_C \tanh z\, dz$, C is the circle $|z| = 2$

(b) $\oint_C \dfrac{dz}{z^3(z+3)}$, C is the contour $|z| = 2$

(c) $\oint_C \dfrac{\cosh z}{z^4}\, dz$, C is the square whose sides are $x = \pm 2$, $y = \pm 2$.

(d) $\oint_C \dfrac{z^2}{z^2 - 3z + (1+i)(2-i)}\, dz$, where C is the curve $x^4 + y^4 = 4$. (Hint: use polar form of the curve to determine whether the singularities are within the contour.)

18. Show that

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$, $|a| < 1$

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$, $|a| > |b|$

(c) $\displaystyle\int_0^{\pi} \frac{d\theta}{1 + \sin^2\theta} = \frac{\pi}{\sqrt{2}}$

(d) $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!\,\pi}{2^{2n}(n!)^2}$, n is a positive integer.

19. Evaluating the following integrals prove that

(a) $\displaystyle\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}\,a^3}$

(b) $\displaystyle\int_0^{\infty} \frac{x^2}{x^6 + 1}\, dx = \frac{\pi}{6}$

(c) $\displaystyle\int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2}\, dx = \frac{\pi}{4b^3}(1 + ab)\, e^{-ab}$, $(a, b > 0)$.

(d) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)}\, dx = \frac{\pi}{(a^2 - b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$, $(a > b > 0)$

(e) $\displaystyle\int_{-\infty}^{+\infty} \frac{x \sin ax}{x^4 + 4}\, dx = \frac{\pi}{2} e^{-a} \sin a$, $(a > 0)$

(f) $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$

(g) $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{1 + e^x}\, dx = \frac{\pi}{\sin \alpha\pi}$, $(|\alpha| < 1)$

(h) $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx = \sec\frac{\alpha}{2}$, $(-\pi < \alpha < \pi)$

(i) $\int_0^\infty \frac{x^{\alpha-1}}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec}, \frac{\alpha\pi}{2} \qquad (0 < \alpha < 2)$

(j) $\int_0^\infty \frac{x^\alpha}{(1+x)^2}\,dx = \frac{\pi\alpha}{\sin\pi\alpha}, \qquad (|\alpha| < 1)$

(k) $\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}$

20. In Example 4, the Laurent series expansion of $f(z) = \frac{1}{z^2(1-z)}$ for $|z| > 1$ was obtained (Eqn 1.72) and it was found that b_1, the coefficient of $1/z$, was equal to zero. $f(z)$ has a simple pole at $z = 1$. Calculate the residue of $f(z)$ at $z = 1$ and explain why this is not equal to b_1.
21. The function $f(z) = \frac{1}{z^2(1-z)}$ has the Laurent series expansion,

$$f(z) = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5}\cdots \qquad \text{(Eqn 1.72)}$$

for $|z| > 1$. Explain why this does not imply that $f(z)$ has an essential singularity at $z = 0$. (Note that $f(z)$ has a pole at $z = 0$).
22. Show that the mapping, $w = \sin z$ transforms the rectangle bounded by $x = \pm\frac{\pi}{2}$, $y = 0$, $y = k$ to an ellipse.
23. Show that the transformation, $w = \frac{z-1}{z+1}$ transforms part of the z-plane to the right of the y-axis to the interior of a circle.
24. If $f(z) = u(x, y) + iv(x, y)$ is analytic, prove that

$$\vec{\nabla} u\cdot\vec{\nabla} v = 0.$$

25. The relation $\vec{\nabla} u\cdot\vec{\nabla} v = 0$ in the preceding problem shows that if the set of curves, u = constant, represents equipotentials then the curves, v = constant, represent the lines of force (or vice-versa).

    An infinite plane is maintained at zero potential and the surface of an infinite cylinder whose axis is parallel to the plane, is maintained at constant potential, V_0. Using the mapping, $z = ia\tan w/2$, (a is real), find the equations for the equipotentials and the lines of force.
26. In Example 15, find the potential to the exterior of the two half cylinders.
27. Use the transformation, $w = z + \frac{1}{z}$ to find the electrostatic potential at any point when a conducting cylinder of unit radius is placed in a uniform electric field with the axis of the cylinder perpendicular to the original direction of the field.
28. Conformal mapping has also applications in hydrodynamics. Again, the motion has to be effectively two-dimensional. Assume that the motion is the same in all planes parallel to the xy–plane and the

velocity is parallel to the plane. For an incompressible fluid having no viscosity, the velocity $\vec{V}$ can be written as $\vec{V} = -\text{ grad } \phi$. The function ϕ, called the *velocity potential*, satisfies:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Suppose $\psi(x, y)$ is a real function such that $\phi(x, y) + i\psi(x, y)$ is analytic in some domain. Then $\psi(x, y)$ is also harmonic. Also, $\vec{\nabla}\phi \cdot \vec{\nabla}\psi = 0$. Thus the velocity $\vec{V}$ is normal to $\vec{\nabla}\psi$ and therefore the tangent to $\psi(x, y) =$ constant at any point gives the direction of velocity. ψ is called the *stream* function.

An infinite cylinder of unit radius is placed in a fluid in uniform motion such that the axis of the cylinder is perpendicular to the original direction of flow. Using the mapping $W = z + \dfrac{1}{z}$, find the stream-lines, equipotentials and velocity at any point. Compare with the previous problem.

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