Del in cylindrical and spherical coordinates

This is a list of some <u>vector calculus</u> formulae for working with common <u>curvilinear</u> coordinate systems.

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Notes

This article uses the standard notation ISO 80000-2, which supersedes ISO 31-11, for spherical coordinates (other sources may reverse the definitions of θ and φ):

- The polar angle is denoted by θ: it is the angle between the z-axis and the radial vector connecting the origin to the point in question.
- The azimuthal angle is denoted by φ : it is the angle between the x-axis and the projection of the radial vector onto the xy-plane.
- The function <u>atan2(y</u>, x) can be used instead of the mathematical function <u>arctan(y/x)</u> owing to its <u>domain</u> and <u>image</u>. The classical arctan function has an image of (-π/2, +π/2), whereas atan2 is defined to have an image of ()()

Coordinate conversions

Conversion between Cartesian, cylindrical, and spherical coordinates^[1] From Cylindrical Cartesian Spherical $x=r\sin\theta\cos\varphi$ x = x $x=\rho\cos\varphi$ Cartesian y = y $y = \rho \sin \varphi$ $y = r \sin \theta \sin \varphi$ z = z $z = r \cos \theta$ z = z $\rho = \sqrt{x^2 + y^2}$ $ho = r \sin heta$ $\rho = \rho$ Cylindrical $\varphi = \varphi$ $\varphi = \varphi$ $\varphi = \arctan\left(\frac{y}{x}\right)$ z = z $z = r \cos \theta$ То z = z $r = \sqrt{x^2 + y^2 + z^2}$ $r=\sqrt{\rho^2+z^2}$ r = r $x^{2} + y^{2}$ $\theta = \theta$ Spherical $\theta = \arctan$ $\theta = \arctan\left(\frac{\rho}{r}\right)$ $\varphi = \varphi$ $\varphi = \varphi$ $\varphi = \arctan\left(\frac{y}{x}\right)$

Unit vector conversions

	Cartesian	Cylindrical	Spherical
Cartesian	N/A		$\begin{aligned} \hat{\mathbf{x}} &= \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} &= \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \end{aligned}$
Cylindrical	$\hat{\rho} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\varphi} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \sin\theta \hat{\mathbf{r}} + \cos\theta \hat{\theta}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\theta}$
Spherical	$\begin{split} \hat{\mathbf{r}} &= \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \\ \hat{\mathbf{\theta}} &= \frac{(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})z - (x^2 + y^2)\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2}} \\ \hat{\mathbf{\phi}} &= \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}} \end{split}$	$\hat{\mathbf{r}} = \frac{\rho \hat{\boldsymbol{\rho}} + z \hat{\mathbf{s}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{z \hat{\boldsymbol{\rho}} - \rho \hat{\mathbf{s}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of source coordinates

	Cartesian	Cylindrical	Spherical
Cartesian	NA	$\hat{\mathbf{x}} = \frac{x\hat{\rho} - y\hat{\varphi}}{\sqrt{a^2 + y^2}}$ $\hat{\mathbf{y}} = \frac{y\hat{\rho} + x\hat{\varphi}}{\sqrt{a^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\begin{split} \hat{\mathbf{x}} &= \frac{x\left(\sqrt{x^2 + y^2} \hat{\mathbf{x}} + z \hat{\boldsymbol{\theta}}\right) - y\sqrt{x^2 + y^2 + z^2} \hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}}\\ \hat{\mathbf{y}} &= \frac{y\left(\sqrt{x^2 + y^2} \hat{\mathbf{x}} + z \hat{\boldsymbol{\theta}}\right) + x\sqrt{x^2 + y^2 + z^2} \hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}}\\ \hat{\mathbf{x}} &= \frac{z \hat{\mathbf{x}} - \sqrt{x^2 + y^2} \hat{\boldsymbol{\theta}}}{\sqrt{x^2 + y^2 + z^2}} \end{split}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \cos \varphi \hat{\mathbf{x}} + \sin \varphi \hat{\mathbf{y}}$ $\hat{\boldsymbol{\varphi}} = -\sin \varphi \hat{\mathbf{x}} + \cos \varphi \hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\rho} = \frac{\rho \hat{\mathbf{r}} + z \hat{\theta}}{\sqrt{\rho^2 + z^2}}$ $\hat{\varphi} = \hat{\varphi}$ $\hat{\mathbf{z}} = \frac{z \hat{\mathbf{r}} - \rho \hat{\theta}}{\sqrt{\rho^2 + z^2}}$
Spherical	$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta \left(\cos\varphi \hat{\mathbf{x}} + \sin\varphi \hat{\mathbf{y}} \right) + \cos\theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta \left(\cos\varphi \hat{\mathbf{x}} + \sin\varphi \hat{\mathbf{y}} \right) - \sin\theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi \hat{\mathbf{x}} + \cos\varphi \hat{\mathbf{y}} \end{aligned}$	$\hat{\mathbf{r}} = \sin \theta \hat{\boldsymbol{\rho}} + \cos \theta \hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos \theta \hat{\boldsymbol{\rho}} - \sin \theta \hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Del formula

		Table with the <u>del</u> operator in cartesian, cylindrical and spheric	cal coordinates
Operation	<u>Cartesian coordinates</u> (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ) , where θ is the polar and φ is the azimuthal angle ^{α}
Vector field A	$A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$	$A_{ ho} \hat{ ho} + A_{arphi} \hat{arphi} + A_{z} \hat{f z}$	$A_r \hat{r} + A_ heta \hat{ heta} + A_arphi \hat{ heta}$
<u>Gradient</u> ∇/ ^[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$rac{\partial f}{\partial ho} \hat{ ho} + rac{1}{ ho} rac{\partial f}{\partial arphi} \hat{arphi} + rac{\partial f}{\partial z} \hat{\mathbf{z}}$	$rac{\partial f}{\partial r}\hat{r}+rac{1}{r}rac{\partial f}{\partial heta}\hat{ heta}+rac{1}{r\sin heta}rac{\partial f}{\partial arphi}\hat{arphi}$
$\underline{\text{Divergence}} \ \nabla \cdot \mathbf{A}^{[1]}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial\left(\rho A_{\rho}\right)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_{\varphi}}{\partial\varphi}+\frac{\partial A_{z}}{\partial z}$	$rac{1}{r^2}rac{\partial\left(r^2A_r ight)}{\partial r}+rac{1}{r\sin heta}rac{\partial}{\partial heta}\left(A_ heta\sin heta ight)+rac{1}{r\sin heta}rac{\partial A_arphi}{\partial arphi}$
$\underline{\textbf{Curt}} \; \boldsymbol{\nabla} \times \mathbf{A}^{[1]}$	$ \begin{pmatrix} \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} \end{pmatrix} \hat{\mathbf{x}} \\ + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\mathbf{y}} \\ + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y} \right) \hat{\mathbf{z}} $	$egin{aligned} &\left(rac{1}{ ho}rac{\partial A_x}{\partial arphi}-rac{\partial A_arphi}{\partial z} ight)\hat{ ho}\ &+\left(rac{\partial A_ ho}{\partial z}-rac{\partial A_z}{\partial ho} ight)\hat{arphi}\ &+rac{1}{ ho}\left(rac{\partial \left(ho A_arphi ight)}{\partial ho}-rac{\partial A_ ho}{\partial arphi} ight)\hat{f z} \end{aligned}$	$rac{1}{r\sin heta}igg(rac{\partial}{\partial heta}(A_arphi\sin heta)-rac{\partial A_ heta}{\partialarphi}igg)\hat{\mathbf{r}}\ +rac{1}{r}igg(rac{1}{\sin heta}rac{\partial A_ au}{\partialarphi}-rac{\partial}{\partial r}(rA_arphi)igg)\hat{ heta}\ +rac{1}{r}igg(rac{\partial}{\partial r}(rA_arphi)-rac{\partial A_ au}{\partial heta}igg)\hat{arphi}$
Laplace operator $\nabla^2 f \equiv \Delta f^{[1]}$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$rac{1}{ ho}rac{\partial}{\partial ho}\left(horac{\partial f}{\partial ho} ight)+rac{1}{ ho^2}rac{\partial^2 f}{\partialarphi^2}+rac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 f}{\partial \varphi^2}$
$\underline{\textbf{Vector Laplacian}} \ \nabla^2 \mathbf{A} \equiv \Delta \mathbf{A}$	$ abla^2 A_x \hat{\mathbf{x}} + abla^2 A_y \hat{\mathbf{y}} + abla^2 A_z \hat{\mathbf{z}}$	$ \begin{pmatrix} \nabla^2 A_{\rho} - \frac{A_{\rho}}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_{\varphi}}{\partial \varphi} \end{pmatrix} \hat{\rho} \\ + \left(\nabla^2 A_{\varphi} - \frac{A_{\varphi}}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_{\varphi}}{\partial \varphi} \right) \hat{\varphi} \\ + \nabla^2 A_{z} \hat{z} $	$\begin{split} & \left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2} \frac{\partial}{\sin\theta} \frac{\partial}{\partial (A_\theta \sin\theta)} - \frac{2}{r^2 \sin\theta} \frac{\partial A_\varphi}{\partial \varphi} \right) \hat{\mathbf{r}} \\ & + \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2}{r^2 \sin^2\theta} \frac{\partial A_\varphi}{\partial \varphi} \right) \hat{\theta} \\ & + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta} \frac{\partial A_r}{\partial \varphi} + \frac{2}{r^2 \sin^2\theta} \frac{\partial A_\theta}{\partial \varphi} \right) \hat{\theta} \end{split}$
$\frac{\text{Material derivative}^{o[2]}}{(A\cdot\nabla)B}$	$\mathbf{A} \cdot \nabla B_z \mathbf{\hat{x}} + \mathbf{A} \cdot \nabla B_y \mathbf{\hat{y}} + \mathbf{A} \cdot \nabla B_z \mathbf{\hat{z}}$	$\begin{split} & \left(A_{\rho}\frac{\partial B_{\rho}}{\partial \rho} + \frac{A_{\varphi}}{\rho}\frac{\partial B_{\rho}}{\partial \varphi} + A_{z}\frac{\partial B_{\rho}}{\partial z} - \frac{A_{\varphi}B_{\varphi}}{\rho}\right)\hat{\rho} \\ & + \left(A_{\rho}\frac{\partial B_{\varphi}}{\partial \rho} + \frac{A_{\varphi}}{\rho}\frac{\partial B_{\varphi}}{\partial \varphi} + A_{z}\frac{\partial B_{\varphi}}{\partial z} + \frac{A_{\varphi}B_{\rho}}{\rho}\right)\hat{\varphi} \\ & + \left(A_{\rho}\frac{\partial B_{z}}{\partial \rho} + \frac{A_{\varphi}}{\rho}\frac{\partial B_{z}}{\partial \varphi} + A_{z}\frac{\partial B_{z}}{\partial z}\right)\hat{z} \end{split}$	$ \begin{array}{c} \left(A_{r}\frac{\partial B_{r}}{\partial r} + \frac{A_{\theta}}{r}\frac{\partial B_{\rho}}{\partial \theta} + \frac{A_{\varphi}}{r\sin\theta}\frac{\partial B_{r}}{\partial \varphi} - \frac{A_{\theta}B_{\theta} + A_{\varphi}B_{\varphi}}{r}\right)\hat{\mathbf{r}} \\ + \left(A_{r}\frac{\partial B_{\theta}}{\partial r} + \frac{A_{\theta}}{r}\frac{\partial B_{\theta}}{\partial \theta} + \frac{A_{\varphi}}{r\sin\theta}\frac{\partial B_{\theta}}{\partial \varphi} + \frac{A_{\theta}B_{r}}{r} - \frac{A_{\varphi}B_{\varphi}\cot\theta}{r}\right)\hat{\mathbf{\rho}} \\ + \left(A_{r}\frac{\partial B_{\varphi}}{\partial r} + \frac{A_{\theta}}{r}\frac{\partial B_{\varphi}}{\partial \theta} + \frac{A_{\varphi}}{r\sin\theta}\frac{\partial B_{\varphi}}{\partial \varphi} + \frac{A_{\varphi}B_{r}}{r} - \frac{A_{\varphi}B_{\varphi}\cot\theta}{r}\right)\hat{\mathbf{\rho}} \end{array} $
Tensor ∇ · T (not confuse with 2nd order tensor divergence)	$- \text{Vew by circling [show]} - \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{xx}}{\partial z} \right) \hat{\mathbf{x}} \\ + \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{xy}}{\partial z} \right) \hat{\mathbf{y}} \\ + \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{xx}}{\partial z} \right) \hat{\mathbf{z}}$	$\begin{split} & \left[\frac{\partial T_{\rho\rho}}{\partial\rho} + \frac{1}{\rho}\frac{\partial T_{\varphi\rho}}{\partial\varphi} + \frac{\partial T_{z\rho}}{\partial z} + \frac{1}{\rho}(T_{\rho\rho} - T_{\varphi\varphi})\right]\hat{\rho} \\ & + \left[\frac{\partial T_{\rho\varphi}}{\partial\rho} + \frac{1}{\rho}\frac{\partial T_{\varphi\varphi}}{\partial\varphi} + \frac{\partial T_{z\varphi}}{\partial z} + \frac{1}{\rho}(T_{\rho\varphi} + T_{\varphi\rho})\right]\hat{\varphi} \\ & + \left[\frac{\partial T_{\rhoz}}{\partial\rho} + \frac{1}{\rho}\frac{\partial T_{\rhoz}}{\partial\varphi} + \frac{\partial T_{zz}}{\partial\varphi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{\rhoz}}{\rho}\right]\hat{z} \end{split}$	$+ \left[\frac{\partial T_{r\theta}}{\partial r} + 2\frac{T_{r\theta}}{r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\cot\theta}{r} T_{\theta\theta} + \frac{1}{r\sin\theta} \frac{\partial T_{\varphi\theta}}{\partial \varphi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta}{r} T_{\varphi\varphi} \right]$
Differential displacement $d \boldsymbol{\ell}^{[1]}$	$dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$	$d ho\hat{ ho}+ hodarphi\hat{arphi}+dz\hat{f z}$	$dr \hat{\mathbf{r}} + r d heta \hat{ heta} + r \sin heta darphi \hat{arphi}$
Differential normal area dS	dy dz x + dx dz ŷ + dx dy ŝ	ρ άφ dz β + dρ dz φ + ρ dρ dφ ŝ	r ² sin θ dθ dφ f + r sin θ dr dφ θ + r dr dθ φ
Differential volume dV ^[1]	dx dy dz	ho d ho darphi dz	$r^2 \sin heta dr d heta darphi$

 $\frac{\Delta \alpha}{\Delta \alpha}$ This page uses θ for the polar angle and φ for the azimuthal angle, which is common notation in physics. The source that is used for these formulae uses θ for the azimuthal angle and φ for the polar angle, which is common mathematical notation. In order to get the mathematics formulae, switch θ and φ in the formulae shown in the table above.

Non-trivial calculation rules

1. div grad $f \equiv \nabla \cdot \nabla f \equiv \nabla^2 f$ 2. curl grad $f \equiv \nabla \times \nabla f = 0$ 3. div curl $\mathbf{A} \equiv \nabla \cdot (\nabla \times \mathbf{A}) = 0$ 4. curl curl $\mathbf{A} \equiv \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ (Lagrange's formula for del) 5. $\nabla^2 (fg) = f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f$

Cartesian derivation

$$\hat{\mathbf{x}}$$

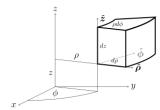
 $\hat{\mathbf{z}}$
 $\hat{\mathbf{z}}$
 dz
 dz
 dx $\hat{\mathbf{y}}$
 dx
 $\hat{\mathbf{y}}$

 $\begin{aligned} \operatorname{div} \mathbf{A} &= \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iint_{V} dV} = \frac{A_{x}(x + dx)dydz - A_{x}(x)dydz + A_{y}(y + dy)dxdz - A_{y}(y)dxdz + A_{z}(z + dz)dxdy - A_{z}(z)dxdy}{dxdydz} \\ &= \frac{\partial A_{x}}{\partial x} + \frac{\partial A_{y}}{\partial y} + \frac{\partial A_{z}}{\partial z} \end{aligned}$

 $(\operatorname{curl} \mathbf{A})_{x} = \lim_{S^{\perp \hat{x}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_{S} dS} = \frac{A_{z}(y + dy)dz - A_{z}(y)dz + A_{y}(z)dy - A_{y}(z + dz)dy}{dydz}$ $= \frac{\partial A_{z}}{\partial y} - \frac{\partial A_{y}}{\partial z}$

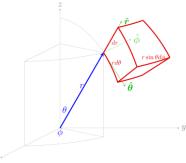
The expressions for $(\operatorname{curl} \mathbf{A})_y$ and $(\operatorname{curl} \mathbf{A})_z$ are found in the same way.

Cylindrical derivation



$$\begin{aligned} \operatorname{div} \mathbf{A} &= \lim_{V \to 0} \frac{\iint_{V \to V} \mathbf{A} \cdot d\mathbf{S}}{\iint_{V \to V}} \\ &= \frac{A_{\rho}(\rho + d\rho)(\rho + d\rho)d\phi dz - A_{\rho}(\rho)\rho d\phi dz + A_{\phi}(\phi + d\phi)d\rho dz - A_{\phi}(\phi)d\rho dz + A_{z}(z + dz)d\rho(\rho + d\rho/2)d\phi - A_{z}(z)d\rho(\rho + d\rho/2)d\phi}{\rho d\phi d\rho dz} \\ &= \frac{1}{\rho} \frac{\partial(\rho A_{\rho})}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_{\phi}}{\partial \phi} + \frac{\partial A_{z}}{\partial z} \\ (\operatorname{curl} \mathbf{A})_{\rho} &= \lim_{S^{J_{\sigma}} - 0} \frac{\int_{\partial S} \mathbf{A} \cdot dt}{\iint_{S} dS} \\ &= \frac{A_{\phi}(z)(\rho + d\rho)d\phi - A_{\phi}(z + dz)(\rho + d\rho)d\phi + A_{z}(\phi + d\phi)dz - A_{z}(\phi)dz}{(\rho + d\rho)d\phi dz} \\ &= -\frac{\partial A_{\phi}}{\partial z} + \frac{1}{\rho} \frac{\partial A_{z}}{\partial \phi} \end{aligned}$$
$$(\operatorname{curl} \mathbf{A})_{\phi} &= \lim_{S^{J_{\sigma}} - 0} \frac{\int_{\partial S} \mathbf{A} \cdot dt}{\iint_{S} dS} \\ &= \frac{A_{z}(\rho)d\rho - A_{z}(\rho + d\rho)dz + A_{\rho}(z + dz)d\rho - A_{\rho}(z)d\rho}{d\rho dz} \\ &= -\frac{\partial A_{z}}{\partial \rho} + \frac{\partial A_{z}}{\partial z} \end{aligned}$$
$$(\operatorname{curl} \mathbf{A})_{z} &= \lim_{S^{J_{\sigma}} - 0} \frac{\int_{\partial S} \mathbf{A} \cdot dt}{\iint_{S} dS} \\ &= \frac{A_{\rho}(\phi)d\rho - A_{\rho}(\phi + d\phi)d\rho + A_{\phi}(\rho + d\rho)(\rho + d\rho)d\phi - A_{\phi}(\rho)\rho d\phi}{\rho d\phi} \\ &= -\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial \phi} + \frac{1}{\rho} \frac{\partial(\rho A_{\rho})}{\partial \rho} \end{aligned}$$
$$\operatorname{curl} \mathbf{A} = (\operatorname{curl} \mathbf{A})_{\rho}\hat{\rho} + (\operatorname{curl} \mathbf{A})_{s}\hat{z} \\ &= \left(\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial \phi} - \frac{1}{\partial Z}\right)\hat{\rho} + \left(\frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_{\rho}}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho} \left(\frac{\partial(\rho A_{\phi})}{\partial \rho} - \frac{\partial A_{\rho}}{\partial \phi}\right)\hat{z} \end{aligned}$$

Spherical derivation



$$\begin{aligned} \operatorname{div} \mathbf{A} &= \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iint_{V} dV} \\ &= \frac{A_{r}(r+dr)(r+dr)d\theta (r+dr) \sin \theta d\phi - A_{r}(r)r d\theta r \sin \theta d\phi + A_{\theta}(\theta+d\theta) \sin(\theta+d\theta) r dr d\phi - A_{\theta}(\theta) \sin(\theta) r dr d\phi + A_{\phi}(\phi+d\phi)(r+dr/2) dr d\theta - A_{\phi}(\phi)(r+dr/2) dr d\theta}{dr r d\theta r \sin \theta d\phi} \end{aligned}$$

$$=rac{1}{r^2}rac{\partial(r^2A_r)}{\partial r}+rac{1}{r\sin heta}rac{\partial(A_ heta\sin heta)}{\partial heta}+rac{1}{r\sin heta}rac{\partial A_\phi}{\partial\phi}$$

1 = $r\sin\theta$

$$(\operatorname{curl} \mathbf{A})_r = \lim_{S^{\perp \hat{r}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_S dS} = \frac{A_{\theta}(\phi) \, r d\theta + A_{\phi}(\theta + d\theta) \, r \sin(\theta + d\theta) d\phi - A_{\theta}(\phi + d\phi) \, r d\theta - A_{\phi}(\theta) \, r \sin(\theta) d\phi}{r d\theta \, r \sin \theta d\phi}$$

$$rac{\partial (A_\phi \sin heta)}{\partial heta} - rac{1}{r \sin heta} rac{\partial A_ heta}{\partial \phi}$$

$$(\operatorname{curl} \mathbf{A})_{\theta} = \lim_{S^{\perp \hat{\theta}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_{S} dS} = \frac{A_{\phi}(r) r \sin \theta d\phi + A_{r}(\phi + d\phi) dr - A_{\phi}(r + dr)(r + dr) \sin \theta d\phi - A_{r}(\phi) dr}{dr r \sin \theta d\phi}$$

$$=\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi}-\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}$$

$$(\operatorname{curl} \mathbf{A})_{\phi} = \lim_{S^{\perp \hat{\phi}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_{S} dS} = \frac{A_{r}(\theta)dr + A_{\theta}(r+dr)(r+dr)d\theta - A_{r}(\theta+d\theta)dr - A_{\theta}(r)rd\theta}{(r+dr/2)drd\theta}$$
$$= \frac{1}{r}\frac{\partial(rA_{\theta})}{\partial r} - \frac{1}{r}\frac{\partial A_{r}}{\partial \theta}$$

$$\operatorname{curl} \mathbf{A} = (\operatorname{curl} \mathbf{A})_r \, \hat{r} + (\operatorname{curl} \mathbf{A})_\theta \, \hat{\theta} + (\operatorname{curl} \mathbf{A})_\phi \, \hat{\phi} = \frac{1}{r \sin \theta} \left(\frac{\partial (A_\phi \sin \theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\phi}$$

Unit vector conversion formula

The unit vector of a coordinate parameter u is defined in such a way that a small positive change in u causes the position vector **v** to change in **u** direction.

Therefore,

$$rac{\partial ec{m{r}}}{\partial u} = rac{\partial s}{\partial u} \hat{m{u}}$$

where s is the arc length parameter.

For two sets of coordinate systems $\boldsymbol{u_i}$ and $\boldsymbol{v_j}$, according to chain rule,

$$d\vec{r} = \sum_{i} \frac{\partial \vec{r}}{\partial u_{i}} du_{i} = \sum_{i} \frac{\partial s}{\partial u_{i}} \hat{u}_{i} du_{i} = \sum_{j} \frac{\partial s}{\partial v_{j}} \hat{v}_{j} dv_{j} = \sum_{j} \frac{\partial s}{\partial v_{j}} \hat{v}_{j} \sum_{i} \frac{\partial v_{j}}{\partial u_{i}} du_{i} = \sum_{i} \sum_{j} \frac{\partial s}{\partial v_{j}} \frac{\partial v_{j}}{\partial u_{i}} \hat{v}_{j} du_{i}$$

Now, let all of *du*_i = 0 but one and then divide both sides by the corresponding differential of that coordinate parameter, we find:

$$\frac{\partial s}{\partial u_i} \hat{u}_i = \sum_j \frac{\partial s}{\partial v_j} \frac{\partial v_j}{\partial u_i} \hat{v}_j$$

See also

Del

- Orthogonal coordinates
- Curvilinear coordinates
- Vector fields in cylindrical and spherical coordinates

References

- 1. Griffiths, David J. (2012). Introduction to Electrodynamics. Pearson. ISBN 978-0-321-85656-2
- 2. Weisstein, Eric W. "Convective Operator" (http://mathworld.wolfram.com/ConvectiveOperator.html). Mathworld. Retrieved 23 March 2011.

External links

Maxima Computer Algebra system scripts (http://www.csulb.edu/~woollett/) to generate some of these operators in cylindrical and spherical coordinates.

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