Del in cylindrical and spherical coordinates

This is a list of some vector calculus formulae for working with common curvilinear coordinate systems.

Contents

Notes

- This article uses the standard notation ISO 80000-2, which supersedes ISO 31-11, for spherical coordinates (other sources may reverse the definitions of θ and φ):
  - The polar angle is denoted by θ: it is the angle between the z-axis and the radial vector connecting the origin to the point in question.
  - The azimuthal angle is denoted by φ: it is the angle between the x-axis and the projection of the radial vector onto the xy-plane.
- The function atan2(y, x) can be used instead of the mathematical function arctan(y/x) owing to its domain and image. The classical arctan function has an image of (−π/2, +π/2), whereas atan2 is defined to have an image of ()()

Coordinate conversions

Conversion between Cartesian, cylindrical, and spherical coordinates[1]

		From		
		Cartesian	**Cylindrical**	**Spherical**
To	**Cartesian**	$x = x$ $y = y$ $z = z$	$x = \rho\cos\varphi$ $y = \rho\sin\varphi$ $z = z$	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$
	Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\varphi = \arctan\left(\frac{y}{x}\right)$ $z = z$	$\rho = \rho$ $\varphi = \varphi$ $z = z$	$\rho = r\sin\theta$ $\varphi = \varphi$ $z = r\cos\theta$
	Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$ $\varphi = \arctan\left(\frac{y}{x}\right)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right)$ $\varphi = \varphi$	$r = r$ $\theta = \theta$ $\varphi = \varphi$

Unit vector conversions

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *destination* coordinates[1]

	Cartesian	**Cylindrical**	**Spherical**
Cartesian	N/A	$\hat{\mathbf{x}} = \cos\varphi\hat{\boldsymbol{\rho}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\varphi\hat{\boldsymbol{\rho}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
Spherical	$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right)z - \left(x^2 + y^2\right)\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$	$\hat{\mathbf{r}} = \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{z\hat{\boldsymbol{\rho}} - \rho\hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of *source* coordinates

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$\hat{\mathbf{x}} = \frac{x\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{y}} = \frac{y\hat{\boldsymbol{\rho}} + x\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \frac{x\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) - y\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$ $\hat{\mathbf{y}} = \frac{y\left(\sqrt{x^2+y^2}\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}\right) + x\sqrt{x^2+y^2+z^2}\hat{\boldsymbol{\varphi}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$ $\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \sqrt{x^2+y^2}\hat{\boldsymbol{\theta}}}{\sqrt{x^2+y^2+z^2}}$
Cylindrical	$\hat{\boldsymbol{\rho}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\boldsymbol{\rho}} = \frac{\rho\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2+z^2}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \frac{z\hat{\mathbf{r}} - \rho\hat{\boldsymbol{\theta}}}{\sqrt{\rho^2+z^2}}$
Spherical	$\hat{\mathbf{r}} = \sin\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\left(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}\right) - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \sin\theta\hat{\boldsymbol{\rho}} + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\hat{\boldsymbol{\rho}} - \sin\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Del formula

Table with the del operator in cartesian, cylindrical and spherical coordinates

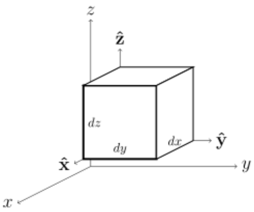
Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar and φ is the azimuthal angle[α]
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\mathbf{A}$[1]	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}}$ $+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}}$ $+\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}}$ $+\frac{1}{\rho}\left(\frac{\partial\left(\rho A_\varphi\right)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}}$ $+\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$
Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	— View by clicking [show] — $\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\rho}}$ $+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$ $+\nabla^2 A_z\hat{\mathbf{z}}$	— View by clicking [show] — $\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial\left(A_\theta\sin\theta\right)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\mathbf{r}}$ $+\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\theta}}$ $+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$
Material derivative[α][2] $(\mathbf{A}\cdot\nabla)\mathbf{B}$	$\mathbf{A}\cdot\nabla B_x\hat{\mathbf{x}} + \mathbf{A}\cdot\nabla B_y\hat{\mathbf{y}} + \mathbf{A}\cdot\nabla B_z\hat{\mathbf{z}}$	$\left(A_\rho\frac{\partial B_\rho}{\partial\rho} + \frac{A_\varphi}{\rho}\frac{\partial B_\rho}{\partial\varphi} + A_z\frac{\partial B_\rho}{\partial z} - \frac{A_\varphi B_\varphi}{\rho}\right)\hat{\boldsymbol{\rho}}$ $+\left(A_\rho\frac{\partial B_\varphi}{\partial\rho} + \frac{A_\varphi}{\rho}\frac{\partial B_\varphi}{\partial\varphi} + A_z\frac{\partial B_\varphi}{\partial z} + \frac{A_\varphi B_\rho}{\rho}\right)\hat{\boldsymbol{\varphi}}$ $+\left(A_\rho\frac{\partial B_z}{\partial\rho} + \frac{A_\varphi}{\rho}\frac{\partial B_z}{\partial\varphi} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}$	— View by clicking [show] — $\left(A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial B_r}{\partial\varphi} - \frac{A_\theta B_\theta + A_\varphi B_\varphi}{r}\right)\hat{\mathbf{r}}$ $+\left(A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial B_\theta}{\partial\varphi} + \frac{A_\theta B_r}{r} - \frac{A_\varphi B_\varphi\cot\theta}{r}\right)\hat{\boldsymbol{\theta}}$ $+\left(A_r\frac{\partial B_\varphi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\varphi}{\partial\theta} + \frac{A_\varphi}{r\sin\theta}\frac{\partial B_\varphi}{\partial\varphi} + \frac{A_\varphi B_r}{r} + \frac{A_\varphi B_\theta\cot\theta}{r}\right)\hat{\boldsymbol{\varphi}}$
Tensor $\nabla\cdot\mathbf{T}$ (not confuse with 2nd order tensor divergence)	— View by clicking [show] — $\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right)\hat{\mathbf{y}}$ $+\left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right)\hat{\mathbf{z}}$	— View by clicking [show] — $\left[\frac{\partial T_{\rho\rho}}{\partial\rho} + \frac{1}{\rho}\frac{\partial T_{\varphi\rho}}{\partial\varphi} + \frac{\partial T_{z\rho}}{\partial z} + \frac{1}{\rho}(T_{\rho\rho} - T_{\varphi\varphi})\right]\hat{\boldsymbol{\rho}}$ $+\left[\frac{\partial T_{\rho\varphi}}{\partial\rho} + \frac{1}{\rho}\frac{\partial T_{\varphi\varphi}}{\partial\varphi} + \frac{\partial T_{z\varphi}}{\partial z} + \frac{1}{\rho}(T_{\rho\varphi} + T_{\varphi\rho})\right]\hat{\boldsymbol{\varphi}}$ $+\left[\frac{\partial T_{\rho z}}{\partial\rho} + \frac{1}{\rho}\frac{\partial T_{\varphi z}}{\partial\varphi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{\rho z}}{\rho}\right]\hat{\mathbf{z}}$	— View by clicking [show] — $\left[\frac{\partial T_{rr}}{\partial r} + 2\frac{T_{rr}}{r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta} + \frac{\cot\theta}{r}T_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi r}}{\partial\varphi} - \frac{1}{r}(T_{\theta\theta} + T_{\varphi\varphi})\right]\hat{\mathbf{r}}$ $+\left[\frac{\partial T_{r\theta}}{\partial r} + 2\frac{T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{\cot\theta}{r}T_{\theta\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\theta}}{\partial\varphi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta}{r}T_{\varphi\varphi}\right]\hat{\boldsymbol{\theta}}$ $+\left[\frac{\partial T_{r\varphi}}{\partial r} + 2\frac{T_{r\varphi}}{r} + \frac{1}{r}\frac{\partial T_{\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial\varphi} + \frac{T_{\varphi r}}{r} + \frac{\cot\theta}{r}(T_{\theta\varphi} + T_{\varphi\theta})\right]\hat{\boldsymbol{\varphi}}$
Differential displacement $d\boldsymbol{\ell}$[1]	$dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$	$d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}}$	$dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,d\varphi\,\hat{\boldsymbol{\varphi}}$
Differential normal area $d\mathbf{S}$	$dy\,dz\,\hat{\mathbf{x}}$ $+\,dx\,dz\,\hat{\mathbf{y}}$ $+\,dx\,dy\,\hat{\mathbf{z}}$	$\rho\,d\varphi\,dz\,\hat{\boldsymbol{\rho}}$ $+\,d\rho\,dz\,\hat{\boldsymbol{\varphi}}$ $+\,\rho\,d\rho\,d\varphi\,\hat{\mathbf{z}}$	$r^2\sin\theta\,d\theta\,d\varphi\,\hat{\mathbf{r}}$ $+\,r\sin\theta\,dr\,d\varphi\,\hat{\boldsymbol{\theta}}$ $+\,r\,dr\,d\theta\,\hat{\boldsymbol{\varphi}}$
Differential volume dV[1]	$dx\,dy\,dz$	$\rho\,d\rho\,d\varphi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\varphi$

^α This page uses θ for the polar angle and φ for the azimuthal angle, which is common notation in physics. The source that is used for these formulae uses θ for the azimuthal angle and φ for the polar angle, which is common mathematical notation. In order to get the mathematics formulae, switch θ and φ in the formulae shown in the table above.

Non-trivial calculation rules

1. $\operatorname{div}\,\operatorname{grad} f \equiv \nabla\cdot\nabla f \equiv \nabla^2 f$
2. $\operatorname{curl}\,\operatorname{grad} f \equiv \nabla\times\nabla f = \mathbf{0}$
3. $\operatorname{div}\,\operatorname{curl}\mathbf{A} \equiv \nabla\cdot(\nabla\times\mathbf{A}) = 0$
4. $\operatorname{curl}\,\operatorname{curl}\mathbf{A} \equiv \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ (Lagrange's formula for del)
5. $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$

Cartesian derivation



$$\operatorname{div}\mathbf{A} = \lim_{V\to 0}\frac{\iint_{\partial V}\mathbf{A}\cdot d\mathbf{S}}{\iiint_V dV} = \frac{A_x(x+dx)dydz - A_x(x)dydz + A_y(y+dy)dxdz - A_y(y)dxdz + A_z(z+dz)dxdy - A_z(z)dxdy}{dxdydz}$$

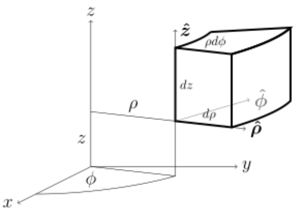
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$(\operatorname{curl}\mathbf{A})_x = \lim_{S^{\perp\hat{\mathbf{x}}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol{\ell}}{\iint_S dS} = \frac{A_z(y+dy)dz - A_z(y)dz + A_y(z)dy - A_y(z+dz)dy}{dydz}$$

$$= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

The expressions for $(\operatorname{curl}\mathbf{A})_y$ and $(\operatorname{curl}\mathbf{A})_z$ are found in the same way.

Cylindrical derivation



$$\operatorname{div}\mathbf{A} = \lim_{V\to 0}\frac{\iint_{\partial V}\mathbf{A}\cdot d\mathbf{S}}{\iiint_V dV}$$

$$= \frac{A_\rho(\rho+d\rho)(\rho+d\rho)d\phi dz - A_\rho(\rho)\rho d\phi dz + A_\phi(\phi+d\phi)d\rho dz - A_\phi(\phi)d\rho dz + A_z(z+dz)d\rho(\rho+d\rho/2)d\phi - A_z(z)d\rho(\rho+d\rho/2)d\phi}{\rho d\phi d\rho dz}$$

$$= \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$(\operatorname{curl}\mathbf{A})_\rho = \lim_{S^{\perp\hat{\boldsymbol{\rho}}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol{\ell}}{\iint_S dS}$$

$$= \frac{A_\phi(z)(\rho+d\rho)d\phi - A_\phi(z+dz)(\rho+d\rho)d\phi + A_z(\phi+d\phi)dz - A_z(\phi)dz}{(\rho+d\rho)d\phi dz}$$

$$= -\frac{\partial A_\phi}{\partial z} + \frac{1}{\rho}\frac{\partial A_z}{\partial\phi}$$

$$(\operatorname{curl}\mathbf{A})_\phi = \lim_{S^{\perp\hat{\boldsymbol{\phi}}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol{\ell}}{\iint_S dS}$$

$$= \frac{A_z(\rho)dz - A_z(\rho+d\rho)dz + A_\rho(z+dz)d\rho - A_\rho(z)d\rho}{d\rho dz}$$

$$= -\frac{\partial A_z}{\partial\rho} + \frac{\partial A_\rho}{\partial z}$$

$$(\operatorname{curl}\mathbf{A})_z = \lim_{S^{\perp\hat{\mathbf{z}}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol{\ell}}{\iint_S dS}$$

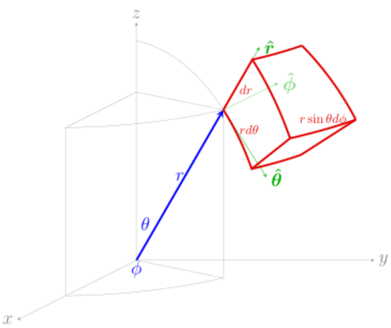
$$= \frac{A_\rho(\phi)d\rho - A_\rho(\phi+d\phi)d\rho + A_\phi(\rho+d\rho)(\rho+d\rho)d\phi - A_\phi(\rho)\rho d\phi}{\rho d\rho d\phi}$$

$$= -\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi} + \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho}$$

$$\operatorname{curl}\mathbf{A} = (\operatorname{curl}\mathbf{A})_\rho\hat{\boldsymbol{\rho}} + (\operatorname{curl}\mathbf{A})_\phi\hat{\boldsymbol{\phi}} + (\operatorname{curl}\mathbf{A})_z\hat{\mathbf{z}}$$

$$= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{z}}$$

Spherical derivation



$$\begin{aligned}\operatorname{div}\mathbf{A} &= \lim_{V\to 0}\frac{\iint_{\partial V}\mathbf{A}\cdot d\mathbf{S}}{\iiint_V dV}\\ &= \frac{A_r(r+dr)(r+dr)d\theta\,(r+dr)\sin\theta d\phi - A_r(r)rd\theta\, r\sin\theta d\phi + A_\theta(\theta+d\theta)\sin(\theta+d\theta)\, rdrd\phi - A_\theta(\theta)\sin(\theta)\, rdrd\phi + A_\phi(\phi+d\phi)(r+dr/2)drd\theta - A_\phi(\phi)(r+dr/2)drd\theta}{dr\, rd\theta\, r\sin\theta d\phi}\\ &= \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\end{aligned}$$

$$\begin{aligned}(\operatorname{curl}\mathbf{A})_r = \lim_{S^{\perp\hat{\boldsymbol r}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol\ell}{\iint_S dS} &= \frac{A_\theta(\phi)\, rd\theta + A_\phi(\theta+d\theta)\, r\sin(\theta+d\theta)d\phi - A_\theta(\phi+d\phi)\, rd\theta - A_\phi(\theta)\, r\sin(\theta)d\phi}{rd\theta\, r\sin\theta d\phi}\\ &= \frac{1}{r\sin\theta}\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\end{aligned}$$

$$\begin{aligned}(\operatorname{curl}\mathbf{A})_\theta = \lim_{S^{\perp\hat{\boldsymbol\theta}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol\ell}{\iint_S dS} &= \frac{A_\phi(r)\, r\sin\theta d\phi + A_r(\phi+d\phi)dr - A_\phi(r+dr)(r+dr)\sin\theta d\phi - A_r(\phi)dr}{dr\, r\sin\theta d\phi}\\ &= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\end{aligned}$$

$$\begin{aligned}(\operatorname{curl}\mathbf{A})_\phi = \lim_{S^{\perp\hat{\boldsymbol\phi}}\to 0}\frac{\int_{\partial S}\mathbf{A}\cdot d\boldsymbol\ell}{\iint_S dS} &= \frac{A_r(\theta)dr + A_\theta(r+dr)(r+dr)d\theta - A_r(\theta+d\theta)dr - A_\theta(r)\, rd\theta}{(r+dr/2)drd\theta}\\ &= \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\end{aligned}$$

$$\operatorname{curl}\mathbf{A} = (\operatorname{curl}\mathbf{A})_r\,\hat{\boldsymbol r} + (\operatorname{curl}\mathbf{A})_\theta\,\hat{\boldsymbol\theta} + (\operatorname{curl}\mathbf{A})_\phi\,\hat{\boldsymbol\phi} = \frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\boldsymbol\theta} + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol\phi}$$

Unit vector conversion formula

The unit vector of a coordinate parameter u is defined in such a way that a small positive change in u causes the position vector $\vec{\boldsymbol r}$ to change in $\hat{\mathbf{u}}$ direction.

Therefore,

$$\frac{\partial\vec{\boldsymbol r}}{\partial u} = \frac{\partial s}{\partial u}\hat{\boldsymbol u}$$

where s is the arc length parameter.

For two sets of coordinate systems u_i and v_j, according to chain rule,

$$d\vec{\boldsymbol r} = \sum_i\frac{\partial\vec{\boldsymbol r}}{\partial u_i}du_i = \sum_i\frac{\partial s}{\partial u_i}\hat{\boldsymbol u}_i du_i = \sum_j\frac{\partial s}{\partial v_j}\hat{\boldsymbol v}_j dv_j = \sum_j\frac{\partial s}{\partial v_j}\hat{\boldsymbol v}_j\sum_i\frac{\partial v_j}{\partial u_i}du_i = \sum_i\sum_j\frac{\partial s}{\partial v_j}\frac{\partial v_j}{\partial u_i}\hat{\boldsymbol v}_j du_i$$

Now, let all of $du_i = 0$ but one and then divide both sides by the corresponding differential of that coordinate parameter, we find:

$$\frac{\partial s}{\partial u_i}\hat{\boldsymbol u}_i = \sum_j\frac{\partial s}{\partial v_j}\frac{\partial v_j}{\partial u_i}\hat{\boldsymbol v}_j$$

See also

- Del
- Orthogonal coordinates
- Curvilinear coordinates
- Vector fields in cylindrical and spherical coordinates

References

1. Griffiths, David J. (2012). *Introduction to Electrodynamics*. Pearson. ISBN 978-0-321-85656-2.
2. Weisstein, Eric W. "Convective Operator" (http://mathworld.wolfram.com/ConvectiveOperator.html). *Mathworld*. Retrieved 23 March 2011.

External links

- Maxima Computer Algebra system scripts (http://www.csulb.edu/~woollett/) to generate some of these operators in cylindrical and spherical coordinates.