Class 9 - Relativistic electrodynamics - I

Class material

Exercise 9.1 - Relativistic invariant expressions of E and B (Jackson 11.14)

(a) Express the Lorentz scalars $F^{\alpha\beta}F_{\alpha\beta}$, $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$ in terms of $\mathbf{E}$ and $\mathbf{B}$. Are there any other invariants quadratic in the fied strengts $\mathbf{E}$ and $\mathbf{B}$?

(b) Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and as a purely magnetic field in some other inertial frame? What are the criteria imposed on $\mathbf{E}$ and $\mathbf{B}$ such that there is an inertial frame in which there is no electric field?

Exercise 9.2 - Motion in combined uniform static electric and magnetic fields (Jackson chapter 12.3)

Consider a charged particle moving in unform and static fields $\mathbf{E}$ and $\mathbf{B}$ in a reference system K.

1. For perpendicular fields with $|\mathbf{E}| < c|\mathbf{B}|$ show that in the reference system K' moving with velocity

$$\mathbf{u} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{1}$$

   the transformed electric field $\mathbf{E}'$ vanishes. From this derive that the particle shows a drift with velocity $\mathbf{u}$ in system K.

2. For perpendicular fields with $c|\mathbf{B}| < |\mathbf{E}|$ show that in the reference system K'' moving with velocity

$$\mathbf{u} = c^2 \frac{\mathbf{E} \times \mathbf{B}}{E^2} \tag{2}$$

   the transformed magnetic field $\mathbf{B}'$ vanishes. From this derive the motion of the particle in system K.

3. Show that when $\mathbf{E}$ and $\mathbf{B}$ are not perpendicular it is impossible to eliminate any of them by changing the reference system.

Exercise 9.3 - Particle drifts in nonuniform, static magnetic fields (Jackson chapter 12.4)

Particles moving in inhomogeneous magnetic fields show characteristic drift phenomena, which are of interest to astrophysical and thermonuclear applications.

1. Assume that the field varies by gradient perpendicular to its direction, i.e. in the direction given by the unit vector $\mathbf{n}$ which satisfies $\mathbf{n} \cdot \mathbf{B} = 0$. Show that in this case to first order of the inhomogeneity the drift velocity is given by

$$\mathbf{v}_G = \omega_B \frac{a^2}{2B^2}(\mathbf{B} \times \nabla_\perp B) \tag{3}$$

   where a is the gyration radius of the particle trajectory in transverse uniform field, ∇ is the gradient in the transverse direction, and $\omega_B = qB/\gamma m$.

2. Assume that the field lines are curved such that the local radius of curvature R is much larger than the gyration radius a. Show that the particle acquires a curvature drift velocity

$$\mathbf{v}_C = \frac{v_\parallel^2}{\omega_B R} \frac{\mathbf{R} \times \mathbf{B}}{R B_0} \tag{4}$$

   where $\mathbf{R}$ is the radius vector from the effective center of curvature to the position of the charge, while $v_\parallel$ is the velocity of the charge parallel to the field lines.

Exercise 9.4 - Propagation of light in moving liquids (Jackson 11.8)

Use the relativistic velocity addition law and the invariance of phase to discuss the Fizeau experiments on the velocity of propagation of light in moving liquids. Show that for liquid flow at a speed v parallel or antiparallel to the path of the light the speed of the light, as observed in the laboratory, is given to first order in v by

$$u = \frac{c}{n(\omega)} \pm v\left(1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{\mathrm{d}n(\omega)}{\mathrm{d}\omega}\right)$$

where ω is the frequency of the light in the laboratory (in the liquid and outside it) and $n(\omega)$ is the index of refraction of the liquid. Because of the extinction theorem, it is assumed that the light travels with speed $u' = c/n(\omega')$ relative to the moving liquid.

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 9.5 - Electric and magnetic fields of a charge in uniform motion*

Using the Lorentz transformation, obtain the

(a) scalar and vector potentials;

(b) electric field strength and magnetic induction

of a charge moving with velocity $\mathbf{v}$, starting from the field of a stationary charge given by the potentials

$$\Phi = \frac{q}{4\pi\epsilon_0 r} \qquad \mathbf{A} = 0\,. \tag{5}$$

and the appropriate electric field strength.

Exercise 9.6 - Velocity selector (Jackson 12.4)*

It is desired to make an $E \times B$ velocity selector with uniform, static, crossed, electric and magnetic fields over a length L. If the entrance and exit slit widths are Δx, discuss the interval Δu of velocities, around the mean value $u = cE/B$, that is transmitted by the device as a function of the mass, the momentum or energy of the incident particles, the field strengths, the length of the selectors, and any other relevant variables. Neglect fringing effects at the ends. Base your discussion on $u \sim 0.5 - 0.995c$. (It is instructive to consider the equation of motion in a frame moving at the mean speed u along the beam direction, as well as in the laboratory.)
Reference: C.A.Coombes *et. al.*, Phys.Rev.**112**, 1303 (1958);
P.Eberhard, M.L.Good, and H.K.Ticho, Rev.Sci.Instrum. **31**, 1054 (1960).

Exercise 9.7 - Trajectory of motion in perpendicular static and uniform electric and magnetic fields (Jackson 12.5)*

A particle of mass m and charge e moves in the laboratory in crossed, static, uniform, electric and magnetic fields. $\mathbf{E}$ is parallel to the x axis; $\mathbf{B}$ is parallel to the y axis.

(a) For $|\mathbf{E}| < |\mathbf{B}|$ make the necessary Lorentz transformation described in 9.2 to obtain explicitly parametric equations for the particle's trajectory.

(b) Repeat the calculation for $|\mathbf{E}| > |\mathbf{B}|$.

These problems are for further practice and to have some fun!

Exercise 9.8 - Trajectory of motion in static and uniform electric and magnetic fields making an angle (Jackson 12.6)

Static, uniform electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, make an angle of θ with respect to each other.

(a) By a suitable choice of axes, solve the force for the motion of a particle of charge e and mass m in rectangular coordinates.

(b) For $\mathbf{E}$ and $\mathbf{B}$ parallel, show that with appropriate constant of integration, etc., the parametric solution can be written

$$x = AR\sin\phi\,, \qquad y = AR\cos\phi\,, \qquad z = \frac{R}{p}\sqrt{1+A^2}\cosh(p\phi)$$
$$ct = \frac{R}{p}\sqrt{1+A^2}\sinh(p\phi)$$

where $R = mc/eB$, $p = E/B$, A is an arbitrary constant, and ϕ is the parameter [actually c/R times the proper time].

Exercise 9.9 - Adiabatic invariance of flux through orbit of particle (Jackson chapter 12.5)

From classical mechanics it is known that for a coordinate q_i which is periodic, the corresponding action integral

$$J_i = \oint p_i dq_i \tag{6}$$

is an adiabatic invariant where p_i is the conjugate momentum. This means if the properties of the system change slowly compared to relevant periods of motion then the action integral J_i is invariant.

Consider now the motion of a particle in a uniform static magnetic field $\mathbf{B}$. The transverse motion is known to be a circular one with frequency $\omega_B = qB/\gamma m$.

(a) Using the expression $\mathbf{p} = m\mathbf{v} + e\mathbf{A}$ for the canonical momentum (where $\mathbf{A}$ is the vector potential corresponding to $\mathbf{B}$, show that the adiabatic invariant corresponding to the transversal motion is essentially the flux $B\pi a^2$ through the particle's orbit.

(b) Show that for a particle moving along direction z in a magnetic field which points mainly to, and has a small positive gradient in the same direction, will eventually be reflected at some point along the z axis (cf. Fig. 1). This allows to construct mirrors for containment of a hot plasma.

Exercise 9.10 - Electrons in the Van Allen belts (Jackson 12.9)

The magnetic field of the Earth can be represented approximately by a magnetic dipole of magnetic moment $M = 8.1 \times 10^{25}$ gauss cm^3.Consider the motion of energetic electrons in the neighborhood of the Earth under the action of this dipole field (Van Allen electron belts). [Note that $\mathbf{M}$ points south].

(a) Show that the equation for a line of magnetic force is $r = r_0 \sin^2\theta$ whete θ is the usual polar angle (colatitude) measured from the axis of the dipole, and find an expression for the magnitude of B along any line of force as a function of θ.

(b) A positively charged particle circles around a line of force in the equatorial plane with a gyration radius a and a mean radius R ($a \ll R$). Show that the particle's azimuthal position (east longitude) changes approximately linearly in time according to

$$\phi(t) = \phi_0 - \frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B(t - t_0)$$

where ω_B is the frequency of gyration at radius R.

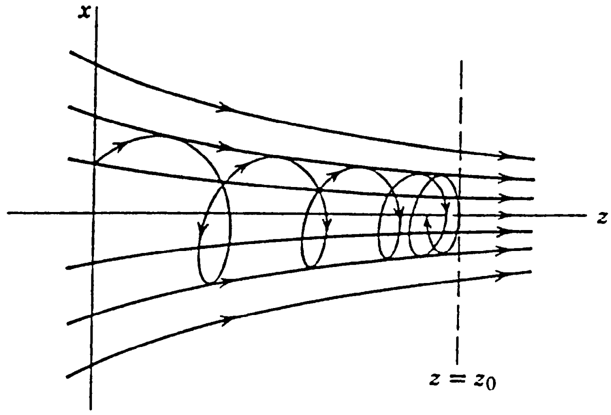

Figure 1: Magnetic mirror

(c) If, in addition to its circular motion of part (b), the particle has a small component of velocity parallel to the lines of force, show that it undergoes small oscillations in θ around $\theta = \pi/2$ with a frequency $\Omega = (3/\sqrt{2})(a/R)\omega_B$. Find the change in longitude per cycle of oscillation in latitude.

(d) For an electron 10 MeV kinetic energy at a mean radius $R = 3 \times 10^7$ m, find ω_B and a, and so determine how long it takes to drift once around the Earth and how long it takes to execute one cycle of oscillation in latitude. Calculate the same quantities for an electron of 10 keV at the same radius.

Exercise 9.11 - Particle in the equatorial plane of Earth (Jackson 12.10)

A charged particle find itself instantaneously in the equatorial plane of the Eath's magnetic field (assumed to be a dipole field) at a distance R from the center of the Earth. Its velocity at that instant makes an angle α with the equatorial plane ($v_{parallel}/v_{\perp} = \tan\alpha$). Assuming that the particle spirals along the lines of force with a gyration radius $a \ll R$, and that flux linked by the orbit is a constant of the motion, find an equation for the maximum magnetic latitude λ reached by the particle as a function of the angle α. Plot the graph (*not a sketch*) of λ versus α. Mark parametrically along the curve the values of α for which a particle at radius R in the equatorial plane will hit the Earth (radius R_0) for $R/R_0 = 1.2,\ 1.5,\ 2.0,\ 2.5,\ 3,\ 4,\ 5$.

Exercise 9.12 - Spin precession (Jackson 12.11)

For this problem you first need to understand the special relativistic theory of Thomas precession, which is the rotation of the instantaneous rest frame of an accelerated object. In case you are not familiar with it yet, see e.g. https://en.wikipedia.org/wiki/Thomas_precession .

Now consider the precession of the spin of a muon, initially longitudinally polarized, as the muon moves in a circular orbit in a plane perpendicular to a uniform magnetic field $\mathbf{B}$.

(a) Show that the difference Ω of the spin precession frequency and the orbital gyration frequency is

$$\Omega = \frac{eBa}{m_\mu}$$

independent of the muon's energy, where $a = (g-2)/2$ is the magnetic moment anomaly. (Find equations of motion for the components of spin along the mutually perpendicular directions defined by the particle's velocity, the radius vector form the center of the circle to the particle, and the magnetic field.)

(b) For the CERN Muon Storage Ring, the orbit radius is $R = 2.5$ meters and $B = 17 \times 10^3$ gauss. What is the momentum of the muon? What is the time dilatation factor γ? How many periods of precession

$T = 2\pi/\Omega$ occur per observed laboratory mean lifetime of the muons? [$m_\mu = 105.66$ MeV, $\tau_0 = 2.2 \times 10^{-6}$ s, $a \approx \alpha/2\pi$].

(c) Express the difference frequency Ω in units of the orbital rotation frequency and compute how many preccessional periods (at the difference frequency) occur per rotation for a 300 MeV muon, a 300 MeV electron, a 5 GeV electron (this last typical of the e^+e^- storage ring at Cornell).