# Class 7 - Skin effect, dispersion, Kramers–Kronig relations

## Class material

### Exercise 7.1 - Skin effect - Current distribution

Consider a current of frequency $\omega$ flowing in a cylindrical conducting wire with radius $R$, conductivity $\sigma$ and magnetic permeability $\mu$. What is the radial distribution of the current?

(a) Write down the Maxwell equations in the quasistationary approximation.

(b) Solve the equation in cylindrical coordinates.

(c) Investigate the current distribution for small and large values of the $\delta$ skin depth.

(d) Compute the dissipated power averaged over one period.

### Exercise 7.2 - Faraday effect 1

Consider a plasma (free electrons) in a homogeneous magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. Assume that a circularly polarised wave of frequency $\omega$ is traveling in the direction of the magnetic field with a circularly polarised electric field $\mathbf{E} = E\mathbf{e}_\pm$ where $\mathbf{e}_\pm = \mathbf{e}_x \pm \imath\mathbf{e}_y$.

(a) Write down the equation of motion of the electrons in the electric field of the wave combined with the background magnetic field.

(b) Solve the equation of motion with the Ansatz $\mathbf{x}(t) = x_0\mathbf{e}_\pm e^{-i\omega t}$ and show that

$$x_0 = \frac{e}{m\omega(\omega \mp \omega_B)}E \tag{1}$$

where $\omega_B = eB_0/m$ is the cyclotron frequency. Show that this leads to a dielectric constant dependent on the circular polarisation

$$\epsilon_\pm = \epsilon_0\left(1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_B)}\right) \tag{2}$$

where $\omega_P$ is the plasma frequency. What are the speeds of propagation $c_\pm$ of the two circular polarisations?

### Exercise 7.3 - Kramers–Kronig relation 1

Use the Kramers–Kronig relation:

$$\mathrm{Re}\,\epsilon(\omega)/\epsilon_0 = 1 + \frac{2}{\pi}\mathcal{P}\int_0^\infty \mathrm{d}\omega' \frac{\omega'\,\mathrm{Im}\,\epsilon(\omega')/\epsilon_0}{\omega'^{\,2} - \omega^2} \tag{3}$$

$$\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 = -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \mathrm{d}\omega' \frac{\mathrm{Re}\,\epsilon(\omega')/\epsilon_0 - 1}{\omega'^{\,2} - \omega^2} \tag{4}$$

to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive $\omega$ as:

$$\mathrm{Im}\,\frac{\epsilon}{\epsilon_0} = \lambda\left[\Theta(\omega - \omega_1) - \Theta(\omega - \omega_2)\right] \quad \text{where} \quad \omega_2 > \omega_1 > 0$$

Sketch the behavior of $\mathrm{Im}\,\epsilon(\omega)$ and the result for $\mathrm{Re}\,\epsilon(\omega)$ as functions of $\omega$. Comment on the reasons for similarities or differences of your results as compared with the curves in the figure showing the dispersion around resonancies. The step function is $\Theta(x) = 0$ if $x < 0$ and $\Theta(x) = 1$ if $x > 0$.

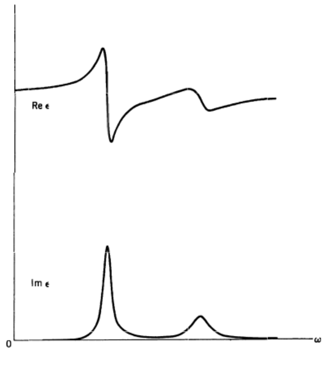

Figure 1

## Exercise 7.4 - Kramers–Kronig relation with static conductivity (Jackson 7.23)

Discuss the extension of the Kramers–Kronig relations (3)and (4) for a medium with a static electrical conductivity $\sigma$. Show that the first equation is unchanged, but the second is changed to

$$\operatorname{Im}\epsilon(\omega) = \frac{\sigma}{\omega} - \frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \mathrm{d}\omega' \frac{\operatorname{Re}\epsilon(\omega') - \epsilon_0}{\omega'^{\,2} - \omega^2}$$

*Hint:* Consider $\epsilon(\omega) - \imath\sigma/\omega$ as analytic for $\operatorname{Im}\omega > 0$.

# Homework

*The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.*

## Exercise 7.5 - Skin effect - surface force (Jackson 8.1)*

Consider the electric and magnetic fields in the surface region of an excellent conductor in the approximation given by:

$$\begin{aligned}\mathbf{E}_c &\approx -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial \mathbf{H}_c}{\partial \xi}\\ \mathbf{H}_c &\approx \frac{\imath}{\mu_c\omega}\mathbf{n}\times\frac{\partial \mathbf{E}_c}{\partial \xi}\end{aligned}$$

what has the solution:

$$\begin{aligned}\mathbf{H}_c &= \mathbf{H}_{\|}\,\mathrm{e}^{-\xi/\delta}\,\mathrm{e}^{\imath\xi/\delta}\\ \mathbf{E}_c &\approx \frac{\mu_c\omega}{2\sigma}(1-\imath)(\mathbf{n}\times\mathbf{H}_{\|})\,\mathrm{e}^{-\xi/\delta}\,\mathrm{e}^{\imath\xi/\delta}\ ,\end{aligned}$$

where the $\delta = \sqrt{2/\mu_c\omega\sigma}$ skin depth is very small compared to the radii of curvature of the surface or the scale of significant spatial variation of the fields just outside, and $\xi$ is the coordinate given by the distance perpendicular to the surface.

(a) For a single-frequency component, show that the magnetic field $\mathbf{H}$ and the current density $\mathbf{J}$ are such that $\mathbf{f}$, the time-averaged force per unit are at the surface from the conduction current, is given by

$$\mathbf{f} = -\mathbf{n}\frac{4}{\mu_c}\left|H_{\|}\right|^2\ ,$$

where $H_{\|}$ is the peak parallel component of magnetic field at the surface, $\mu_c$ is the magnetic permeability of the conductor, and $\mathbf{n}$ is the outward normal at the surface.

(b) If the magnetic permeability $\mu$ outside the surface is different from $\mu_c$, is there an additional magnetic force per unit area? What about electric forces?

(c) Assume that the fields are a superposition of different frequencies (all high enough that the approximations still hold). Show that the time-averaged force takes the same form as in part (a) with $\left|H_{\|}\right|^2$ replaced by $2\langle\left|H_{\|}\right|^2\rangle$, where the angle brackets $\langle\ldots\rangle$ mean time average.

## Exercise 7.6 - Faraday effect 2*

Consider a plasma (free electrons) in a homogeneous magnetic field $\vec{B} = B_0\mathbf{e}_z$. Assume that a circularly polarised wave of frequency $\omega$ is traveling in the direction of the magnetic field with a linearly polarised electric field $\mathbf{E} = E\mathbf{e}_x$ over a distance $l$.

(a) Using the results of 7.2 show that the polarisation direction is rotated by an angle $\Delta\varphi$ which is proportional to the distance $l$.

(b) Assuming that $B_0$ is small enough, expand to first order to obtain

$$\Delta\varphi = \mathcal{V}B_0 l \tag{5}$$

What is the value of the Verdet constant $\mathcal{V}$?

## Exercise 7.7 - Kramers–Kronig relation 2*

Use the Kramers–Kronig relation to calculate the real part of $\epsilon(\omega)$, given the im e $\omega$ as:

$$\operatorname{Im}\frac{\epsilon}{\epsilon_0} = \lambda\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

Sketch the behavior of $\operatorname{Im}\epsilon(\omega)$ and the result for $\operatorname{Re}\epsilon(\omega)$ as functions of $\omega$. Comment on the reasons for similarities or differences of your results as compared with the curves in the figure showing the dispersion around resonancies.

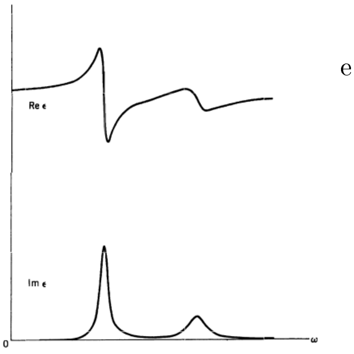

Figure 2

*These problems are for further practice and to have some fun!*

## Exercise 7.8 - Lorentz model

Consider the harmonicaly bound electron model, which is known as the Lorentz model.

(a) Compute the dielectric constant and the reflection coefficient.

(b) Determine the relation between the conductivity and the dielectric constant from the definition of the polarization current.

(c) Discuss the cases of the free and the damped electron gas.

## Exercise 7.9 - Energy loss of a charged particle in a medium (Jackson 7.26)

A charged particle (charge $Ze$) moves at constant velocity $\mathbf{v}$ through a medium described by a dielectric function $\epsilon(\mathbf{q},\omega)/\epsilon_0$ or, equivalently, by a conductivity function $\sigma(\mathbf{q},\omega) = \imath\omega\left[\epsilon_0-\epsilon(\mathbf{q},\omega)\right]$. It is desired to calculate the energy loss per unit time by the moving particle in terms of the dielectric function $\epsilon(\mathbf{q},\omega)$ in the approximation that the electric field is the negative gradient of the potential and the current flow obeys Ohm's law, $\mathbf{J}(\mathbf{q},\omega) = \sigma(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$.

(a) Show that with suitable normalization, the Fourier transform of the particle's charge density is:

$$\rho(\mathbf{q},\omega) = \frac{Ze}{(2\pi)^3}\delta(\omega-\mathbf{qv})$$

(b) Show that the Fourier components of the scalar potential are:

$$\phi(\mathbf{q},\omega) = \frac{\rho(\mathbf{q},\omega)}{q^2\epsilon(\mathbf{q},\omega)}$$

(c) Starting from

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int \mathrm{d}^3x\mathbf{JE}$$

show that the energy loss per unit time can be written as

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{Z^2e^2}{4\pi^3}\int\frac{\mathrm{d}^3q}{q^2}\int_0^\infty \mathrm{d}\omega\omega\operatorname{Im}\left[\frac{1}{\epsilon(\mathbf{q},\omega)}\right]\delta(\omega-\mathbf{qv})$$

This shows that $\mathrm{Im}\left[\epsilon(\mathbf{q},\omega)\right]^{-1}$ is related to energy loss and provides, by studying characteristic energy losses in thin foils, information on $\epsilon(\mathbf{q},\omega)$ for solids.