Class 4 - Applications of the magnetic scalar and vector potential. Dielectrics and polarisation charges

Class material

Exercise 4.1 - Quarter-spheres with alternating potentials

Consider a metal sphere shell with radii a, with its centre at the origin. We cut the sphere shell to four segments with the xz and yz planes, and insulate the segments from each other. Then the fur segments ar placed at potentials $\pm V_0$ in an alternating fashion as shown on Fig.1.

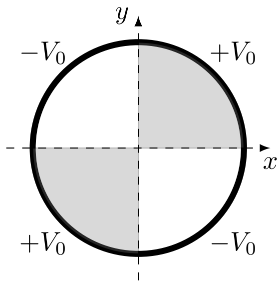

Figure 1

(a) Compute the $\Phi_{\text{out}}(\vec{r})$ electrostatic potential outside the sphere!

(b) Compute the $\Phi_{\text{in}}(\vec{r})$ electrostatic potential inside the sphere!

(c) Compute the $\vec{E}(z)$ electric field along the full z axis!

Exercise 4.2 - Magnetic field of a sphere magnet

Consider a spherical permanent magnet with radius R and homogenous magnetisation M. Determine the magnetic field and the magnetic induction in the whole space!

(a) Write down the equations and the boundary conditions satisfied by the magnetic scalar potential!

(b) Determine the general solution in spherical coordinates!

(c) Determine the expansion coefficients from the boundary conditions!

Exercise 4.3 - Magnetic field of a rotated charged sphere

A sphere with radius R and uniform surface charge density σ is rotating with angular velocity ω. What are the resulting magnetic field and magnetic induction?

(a) We can describe the magnetic field with the magnetic scalar potential since there is current onéy on the shell of the sphere. Write the equations and the boundary conditions for the magnetic scalar potential!

(b) Give the general solution in spherical coordinates!

(c) Determine the expansion coefficients from the boundary conditions!

Exercise 4.4 - Dielectric sphere in homogenous electric field

We put a sphere with radius R and dielectric constant ϵ into homogen external electric field with magnitude E_0. What are the resulting electric field and the surface charge density?

(a) Write the corresponding Laplace equation and the boundary conditions!

(b) Recognise the analogy with the magnetized sphere from Exercise 3.2 of the previous class!

(c) Give the formed electric field and the surface plarisation charge density!

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 4.5 - Surface charge density on dielectric sphere in external field*

Based on the result we obtained in class:

$$\Phi_{\text{in}}(\vec{r}) = -E_0 \frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} r \cos\theta$$
$$\Phi_{\text{out}}(\vec{r}) = -E_0 r \cos\theta + E_0 \frac{3\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \frac{\cos\theta}{r^2}$$

give the polarisation charge density on the surface of the sphere from Gauss's law.

Exercise 4.6 - Electric potential of semi-spheres with different surface charge density*

Consider a sphere shell around the origin with radius a. The $z > 0$ semi-sphere is charged with $+\sigma_0$ while the $z < 0$ with $-\sigma_0$ surface charge density. Inside the sphere denote the electrostatic potential with $\Phi_{\text{in}}(\vec{r})$ and outside with $\Phi_{\text{out}}(\vec{r})$.

(a) Write the general form of the potential inside and outside the sphere!

(b) Give the corresponding boundary conditions!

(c) Determine the approximate $\Phi_{\text{in}}(\vec{r})$ and $\Phi_{\text{out}}(\vec{r})$ functions up to 6th order!

Exercise 4.7 - Magnetic field of a rotated charged disk*

On the upper surface of a tiny disc with radius a there is $\sigma > 0$ surface charge density and it rotates around the z axis with angular velocity ω_z.

(a) Compute the magnetic field $B(z)$ along the z axis if the centre of the disc is at the origin!

(b) Determine the magnetic scalar potential $\Psi_m(z)$ along the z axis!

(c) Draw the functions $B(z)$ and $\Psi_m(z)$!

(d) Determine the potential $\Psi_m^+(z)$ for $r \gg a$ using the expansion technique we learned in class!

(e) Determine the potential $\Psi_m^+(z)$ for $r \ll a$ using the expansion technique we learned in class!

(f) Determine the vector potential $\vec{A}(\vec{r})$ in the plane of the disc near the centre using $\Psi_m^-(z)$!

These problems are for further practice and to have some fun!

Exercise 4.8 - Rotating magnetized sphere (Jackson 6.4)

A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi MR^3/3$ rotates about its magnetization axis with angular speed ω. In the steady state no current flows in the conductor. The motion is nonrelativistic; the sphere has no excess charge on it.

(a) From the fact that in stationary case the force acting on the electrons is zero, show that the motion induces an electric field and a uniform volume charge density in the conductor, $\rho = -m\omega/\pi c^2 R^3$. Use the magnetic field derived on class:

$$\vec{B}_{\text{in}} = \frac{2\mu_0}{3}\vec{M}$$

(b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exist outside and that the quadrupole moment tensor has nonvanishing components, $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = -Q_{33}/2$.

First integrate the electic field determined inside in order to find the electric potential inside, then express the result on Legendre polinomials and from that fit the boundary conditions of the outside potential. Which is the only non-vanishing term in the outside potential?

Which quadrupole term is it?

(c) By considering the radial electric field inside and outside the sphere, show that the necessary surface-charge density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2} \cdot \frac{4m\omega}{3c^2} \cdot \left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

(d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact (by any path) is $\mathcal{E} = \mu_0 m\omega/4\pi R$.

[See Landau and Lifshitz, *Electrodynamics of Continuous Media*, p.221, for an alternative discussion of this eletromotive force.]

Exercise 4.9 - Finite closed coaxial cylinders with different potentials on the walls

Consider two coaxial cylinder walls along the z axis with radii R_1 and R_2, with height h. The top and bottom opening between the cylinders is covered with an annulus, resulting a closed torus shaped box, with the bottom of the box at $z = 0$ and the top at $z = h$. We the bottom and the top annulus are grounded, while the inner wall is kept at potential V_1 and the other one at potential V_2.

(a) Working in cylindrical coordinate system, write down the general form of the potential function $\Phi(\vec{r})$ inside the torus!

(b) Give the corresponding boundary conditions!

(c) Determine the $\Phi(\vec{r})$ potential function from the boundary conditions!

Exercise 4.10 - The spherical form of $\nabla \times (\nabla \times\ )$

Consider a sphere shell around the origin with radii R and surface charge density $\sigma_0 > 0$, rotating around the z axis with angilar velocity ω_0. The magnetic scalar potentials derived in the class are:

$$\Phi_{\text{in}} = \frac{-2Mr\cos\theta}{3}$$
$$\Phi_{\text{out}} = \frac{MR^3\cos\theta}{3}$$

where $M = \sigma\omega R$, and the magnetic field is $\vec{B} = -\mu_0 \vec{\nabla}\Phi$

(a) Determine the (simplest) vector potential $\vec{A}(\vec{r})$ from the field $\vec{B}(\vec{r})$!

(b) In magnetostatics, the vector potential satisfies the following equation:

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{j} \,.$$

Write this equation in spherical coordinates!

(c) Check that the previous result obtained for the vector potential satisfies this equation!

Exercise 4.11 - Conducting cylinder with a sphere inside

Consider a cylinder with height b and cross-section A, filled with homogeneous matter with conductivity σ_1, except for the middle of the cylinder where there is a spherical region filled with a homogeneous matter with conductivity σ_2. The a parameter is much smaller than any geometric parameter of the cylinder.

(a) Give the R electric resistance of the cylinder approximately!

(b) We give U voltage to the R resistance. Determine the maximal current and the maximal power density!

Hint: recall the analogy between electrostatic potential and static current problems we learned in the Electrodynamics 1 course.

Exercise 4.12 - Magnetic screening (Jackson 5.14)

A long hollow, right circular cylinder of inner (outer) radius $a(b)$, and of relatice permeability μ_r, is placed in a region of initially uniform magnetic-flux density $\mathbf{B}_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of $\mathbf{B}$ on the cylinder axis to $\mathbf{B}_0$ as a function of $\log_{10}\mu_r$ for $a^2/b^2 = 0.5, 0.1$. Neglect end effects.

Exercise 4.13 - Magnetic scalar potential of two parallel wires (Jackson 5.15)

Consider two long, straight wires, parallel to the z axis, spaced a distance d apart and carrying currents I in opposite directions. Describe the magnetic field $\mathbf{H}$ in terms of a magnetic scalar potential Φ_M, with $\Phi_M = -\nabla$.

(a) If the wires are parallel to the z axis with positions, $x = \pm d/2$, $y = 0$, show that in the limit of small spacing, the potential is approximately that of a two dimensional dipole,

$$\Phi_M \simeq -\frac{Id\sin\phi}{2\pi\rho} + \mathcal{O}(d^2/\rho^2)$$

where ρ and ϕ are the usual polar coordinates.

(b) The closely spaced wires are now centered in a hollow right circular cylinder of steel, of inner (outer) radius $a(b)$ and magnetic permeability $\mu = \mu_0\mu_r$. Determine the magnetic scalar potential in the three regions, $0 < \rho < a$, $a < \rho < b$, and $b < \rho$. Show that the field outside the steel cylinder is a two-dimensional dipole field, as in part (a), but with a strength reduced by the factor

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}$$

Relate your result to Exercise 4.12.

(c) Assuming that $\mu_r \gg 1$, and $b = a + t$, where the thickness $t \ll b$, write down an approximate expression for F and determine its numerical value for $\mu_r = 200$ (typical of stell at 20 G), $b = 1.25$ cm, $t = 3$ nm. The shielding effect is relevant for reduction of stray fields in residential and commerical 60 Hz, 110 or 220 V wieing. The figure illustrates the shielding effect for $a/b = 0.9$, $\mu_r = 100$.

Exercise 4.14 - Magnetic scalar potential of a surface current on a cylinder (Jackson 5.30)

(a) Show that a surface current density $K(\phi) = I\cos\phi/2R$ flowing in the axial direction on a right circular cylindrical surface of radius R produces inside the cylinder a uniform magnetic induction $B_0 = \mu_0 I/4R$ in a direction perpendicular to the cylinder axis. Show that the field outside is that of a two-dimensional dipole.

(b) Calculate the total magnetistatic field energy per unit length. How is it divided inside and outside the cylinder?

(c) What is the inductance per unit length of the system, viewed as a long circuit with current flowing up one side of the cylinder and back the other? $L = \pi\mu_0/8$

Exercise 4.15 - Magnetic scalar potential of a rotating dieletric sphere (Jackson 6.8)

A dielectric sphere of dielectric constant ϵ and radius a is located at the origin. There is a uniform applied electric fied E_0 in the x direction. The sphere rotates with an angular velocity ω about the z axis. Show that there is a magnetic field $\mathbf{H} = \nabla\Phi_M$, where

$$\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega \left(\frac{a}{r_>}\right)^5 xz$$

where $r_>$ is the larger of r and a. The motion is nonrelativistic.

You can use the results for the surface charge density of the dielectric sphere in an applied field

$$\sigma_{\text{pol}} = 3\epsilon_0\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)E_0\cos\theta$$

Exercise 4.16 - Magnetic vector potential (Jackson 5.8)

A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r and θ (or ρ and z): $\mathbf{J} = \hat{\phi}J(r,\theta)$. The distribution is 'hollow' in the sense that there is a current-free region near the origin, as well as outside.

(a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L m_L r^L P_L^1(\cos\theta)$$

in the interior and

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L \mu_L r^{-L} P_L^1(\cos\theta)$$

(b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)}\int \mathrm{d}^3x r^{-L-1} P_L^1(\cos\theta) J(r,\theta)$$

and

$$m_L = -\frac{1}{L(L+1)}\int \mathrm{d}^3x r^{L} P_L^1(\cos\theta) J(r,\theta)$$

Exercise 4.17 - Magnetic vector potential (Jackson 5.9)

Two circular coils of radius a and separation b can be described in cylindrical coordinates by the current density

$$\mathbf{J} = \hat{\phi}I\delta(\rho-a)\left[\delta(z-b/2)+\delta(z+b/2)\right]$$

(a) Using the formalism of Exercise 4.16, calculate the internal and external multipole moments for $L = 1, \ldots, 5$.

(b) Using the internal multipole expansion of Exercise 4.16, write down explicitly an expression for B_z on the z axis.